Musterlösung zur Aufgabe 11. Freie Elektronen in d Dimensionen und die Sommerfeld-Entwicklung (20 Punkte)

$$n = \int_{-\infty}^{+\infty} d\varepsilon \rho(\varepsilon) f_\mu(\varepsilon) = \int_{-\infty}^{\mu} d\varepsilon \rho(\varepsilon) \; + a_1 \rho'(\mu)(k_B T)^2 + a_2 \rho'''(\mu)(k_B T)^4 + \mathcal{O}(T^6)$$
$$= n_0 + \int_{\mu_0}^{\mu} d\varepsilon [\rho(\mu_0) + (\varepsilon - \mu_0)\rho'(\mu_0) + \ldots] + a_1 \rho'(\mu)(k_B T)^2 + a_2 \rho'''(\mu)(k_B T)^4 + \mathcal{O}(T^6)$$
$$= n_0 + \rho(\mu_0)(\mu - \mu_0) + \frac{1}{2}\rho'(\mu_0)(\mu - \mu_0)^2 + a_1 \left.\frac{d}{d\varepsilon}[\rho(\varepsilon)]\right|_{\varepsilon=\mu} (k_B T)^2 + a_2 \rho'''(\mu)(k_B T)^4 + \mathcal{O}(T^6)$$

$$\left.\frac{d}{d\varepsilon}[\rho(\varepsilon)]\right|_{\varepsilon=\mu} = \left.\frac{d}{d\varepsilon}\left[\rho(\mu_0) + (\varepsilon - \mu_0)\rho'(\mu_0) + \frac{1}{2}(\varepsilon - \mu_0)^2 \rho''(\mu_0) + \ldots\right]\right|_{\varepsilon=\mu} = \rho'(\mu_0) + (\mu - \mu_0)\rho''(\mu_0) + \ldots$$

$$0 \overset{!}{=} n(T) - n_0 = \rho(\mu_0)\Delta\mu + \frac{1}{2}\rho'(\mu_0)\Delta\mu^2 + a_1 \left[\rho'(\mu_0) + \rho''(\mu_0)\Delta\mu\right](k_B T)^2 + a_2 \rho'''(\mu_0)(k_B T)^4 + \mathcal{O}(T^6)$$

Potenzreihenansatz: $\Delta\mu = b_2 (k_B T)^2 + b_4 (k_B T)^4$; $x \equiv k_B T$

$$0 = \rho(\mu_0)(b_2 x^2 + b_4 x^4) + \frac{1}{2}\rho'(\mu_0) b_2^2 x^4 + a_1 [\rho'(\mu_0) + \rho''(\mu_0) b_2 x^2] x^2 + a_2 \rho'''(\mu_0) x^4$$

$$0 = x^2(\rho_0 b_2 + \rho_0' a_1) + x^4(\rho_0 b_4 + \frac{1}{2}\rho_0' b_2^2 + \rho_0'' a_1 b_2 + \rho_0''' a_2)$$

$$\Rightarrow \; b_2 = -\frac{a_1 \rho_0'}{\rho_0}; \; b_4 = -\frac{\frac{1}{2}\rho'(\mu_0) b_2^2 + a_1 \rho_0'' b_2 + a_2 \rho_0'''}{\rho_0}$$

Now apply to free particles: $\rho(\varepsilon) = A_d \, \varepsilon^{\frac{d}{2}-1}$

$$\Rightarrow \; \rho'(\varepsilon) = \left(\frac{d}{2} - 1\right)\frac{\rho(\varepsilon)}{\varepsilon}; \; \rho''(\varepsilon) = \frac{(d-2)(d-4)}{4}\frac{\rho(\varepsilon)}{\varepsilon^2}; \; \rho'''(\varepsilon) = \frac{(d-2)(d-4)(d-6)}{8}\frac{\rho(\varepsilon)}{\varepsilon^3}.$$

$$\overset{d=3}{\Rightarrow} \; b_2 = -\frac{a_1}{2\varepsilon_F}; \; b_4 = -\left[\frac{1}{4\varepsilon_F}\frac{a_1^2}{4\varepsilon_F^2} + \frac{a_1}{4\varepsilon_F^2}\frac{a_1}{2\varepsilon_F} + \frac{3a_2}{8\varepsilon_F^3}\right] = -\frac{3}{8\varepsilon_F^3}\left[a_2 + \frac{1}{2}a_1^2\right]$$

Now energy density:

$$U = U_0 + \int_{\mu_0}^{\mu} d\varepsilon \, \varepsilon\rho(\varepsilon) + a_1 \left.\frac{d}{d\varepsilon}(\varepsilon\rho(\varepsilon))\right|_{\varepsilon=\mu} x^2 + a_2 \left.\frac{d^3}{d\varepsilon^3}(\varepsilon\rho(\varepsilon))\right|_{\varepsilon=\mu} x^4$$

$$\tilde{\rho}(\varepsilon) \equiv \varepsilon\rho(\varepsilon)$$

$$U - U_0 = x^2(\tilde{\rho}_0 b_2 + a_1 \tilde{\rho}_0') + x^4(\tilde{\rho}_0 b_4 + \frac{1}{2}\tilde{\rho}_0' b_2^2 + a_1 \tilde{\rho}_0'' b_2 + a_2 \tilde{\rho}_0''')$$

Free particles: $\tilde{\rho}(\varepsilon) = A_d\, \varepsilon^{\frac{d}{2}}$

$$\Rightarrow\ \tilde{\rho}'(\varepsilon) = \frac{d}{2}\frac{\tilde{\rho}(\varepsilon)}{\varepsilon};\ \tilde{\rho}''(\varepsilon) = \frac{d(d-2)}{4}\frac{\tilde{\rho}(\varepsilon)}{\varepsilon^2};\ \tilde{\rho}'''(\varepsilon) = \frac{d(d-2)(d-4)}{8}\frac{\tilde{\rho}(\varepsilon)}{\varepsilon^3}.$$

$$\begin{aligned}
\overset{d=3}{\Rightarrow}\ U - U_0 &= x^2\tilde{\rho}_0\left(b_2 + \frac{3a_1}{2\varepsilon_F}\right) + x^4\tilde{\rho}_0\left(b_4 + \frac{1}{2}b_2^2\frac{3}{2\varepsilon_F} + \frac{3}{4}\frac{a_1}{\varepsilon_F^2}b_2 - \frac{3}{8}\frac{a_2}{\varepsilon_F^3}\right)\\
&= a_1\rho_0 x^2 + x^4\frac{\rho_0}{\varepsilon_F^2}\left[-\frac{3}{8}\left(a_2 + \frac{1}{2}a_1^2\right) + \frac{3}{16}a_1^2 - \frac{3}{8}a_1^2 - \frac{3}{8}a_2\right]\\
&= a_1\rho_0 x^2 - \frac{3}{8\varepsilon_F^2}(2a_2 + a_1^2)\rho_0 x^4
\end{aligned}$$

$$\Rightarrow\ c_{V,N} = 2a_1\rho_0 x - \frac{3}{2\varepsilon_F^2}(2a_2 + a_1^2)\rho_0 x^3 + \mathcal{O}(x^5)$$

$d = 2$: $b_2 = b_4 = 0$ $\tilde{\rho}_0' = \rho_0$; $\tilde{\rho}_0'' = \tilde{\rho}_0''' = 0$

$U - U_0 = a_1\rho_0 x^2 \Rightarrow c_{V,N} = 2a_1\rho_0 x$