Reminder: scaling equations

$$M(L,T) = \tilde{M}(\varepsilon L^{1/\nu})\, L^{-\beta/\nu}$$

$$\chi(L,T) = \tilde{\chi}(\varepsilon L^{1/\nu})\, L^{\gamma/\nu}$$

$$C(L,T) = \tilde{C}(\varepsilon L^{1/\nu})\, L^{\alpha/\nu}$$

The functions $\tilde{M}$, $\tilde{\chi}$, $\tilde{C}$ etc. can be expressed by derivatives of the universal function Y and the nonuniversal coefficients C_1, C_2.

Note: (i) The scaling laws are valid only asymptotically for $\varepsilon \to 0$ and $B \to 0$; in practice, deviations (corrections to scaling) can be significant.
(ii) In general, there are additional regular contributions.
(iii) Different analysis needed for 1^{st} order transitions.

Practical application: in order to test critical exponents and to determine nonuniversal exponents, one would have to plot scaled observables against scaled parameters, e.g. $\tilde{M}_L(x)$, where $\tilde{M}_L = M_L L^{\beta/\nu}$; $x = \varepsilon L^{1/\nu}$.

The scaled curves should collapse asymptotically:

$$\tilde{M}(x) = \lim_{L\to\infty} (x) \quad \text{(separately for } T<T_c \text{ and } T>T_c\text{)}.$$

Problem: (i) The mapping $T \to \varepsilon = |1 - \frac{T}{T_c}|$ requires the knowledge of T_c (nonuniversal). (ii) The exponents are in general unknown or only approximately known.

Solution: Determine T_c from observables which are rescaled only via the argument x, i.e. with trivial renormalizing prefactor $L^0 = 1$. Example: from

$$\langle m^2 \rangle \propto \left(L^d L^{-\beta/\nu}\right)^2 \quad \text{and} \quad \langle m^4 \rangle \propto \left(L^d L^{-\beta/\nu}\right)^4$$

$m = L^d M$

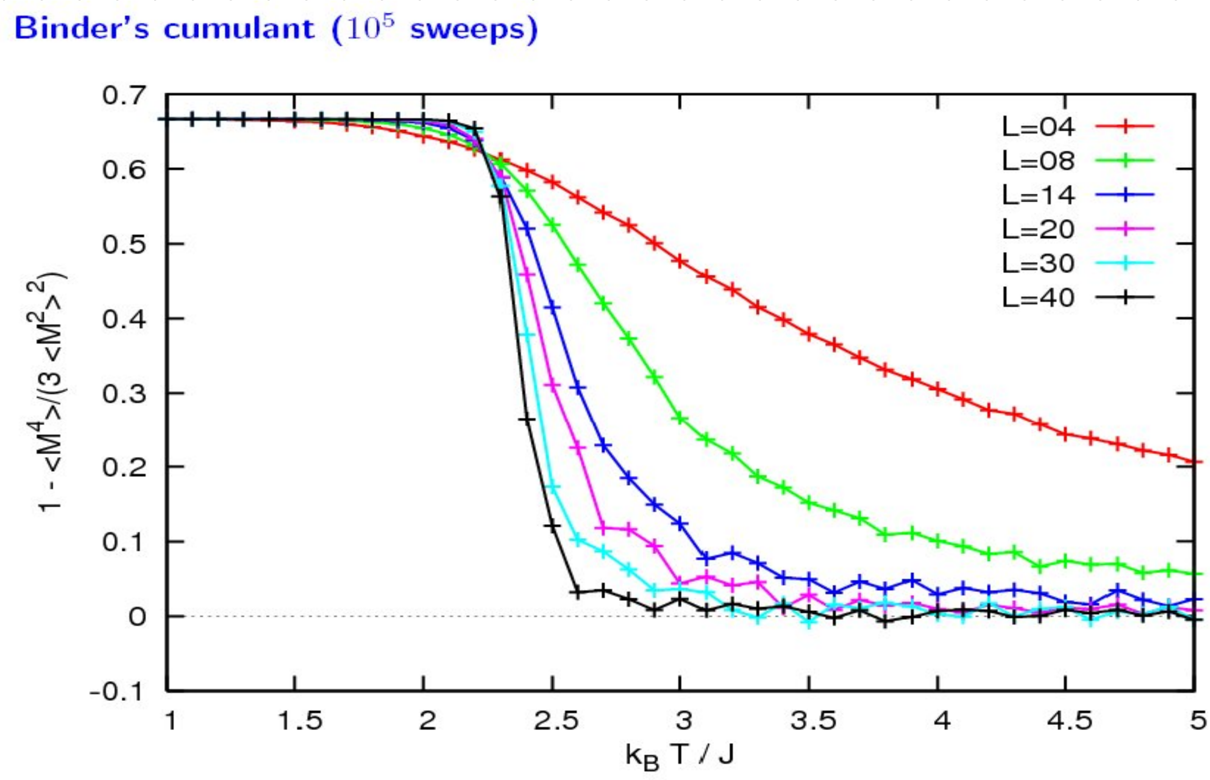
construct scaling-free ratio $\frac{\langle m^4 \rangle}{\langle m^2 \rangle^2} \propto \left(L^{d-\frac{\beta}{\nu}}\right)^{4-2\cdot 2} = L^0$

or, equivalently, the Binder cumulant $U_4 = 1 - \frac{\langle m^4 \rangle}{3\langle m^2 \rangle}$

with $U_4(L,T) = \tilde{U}_4(x)$

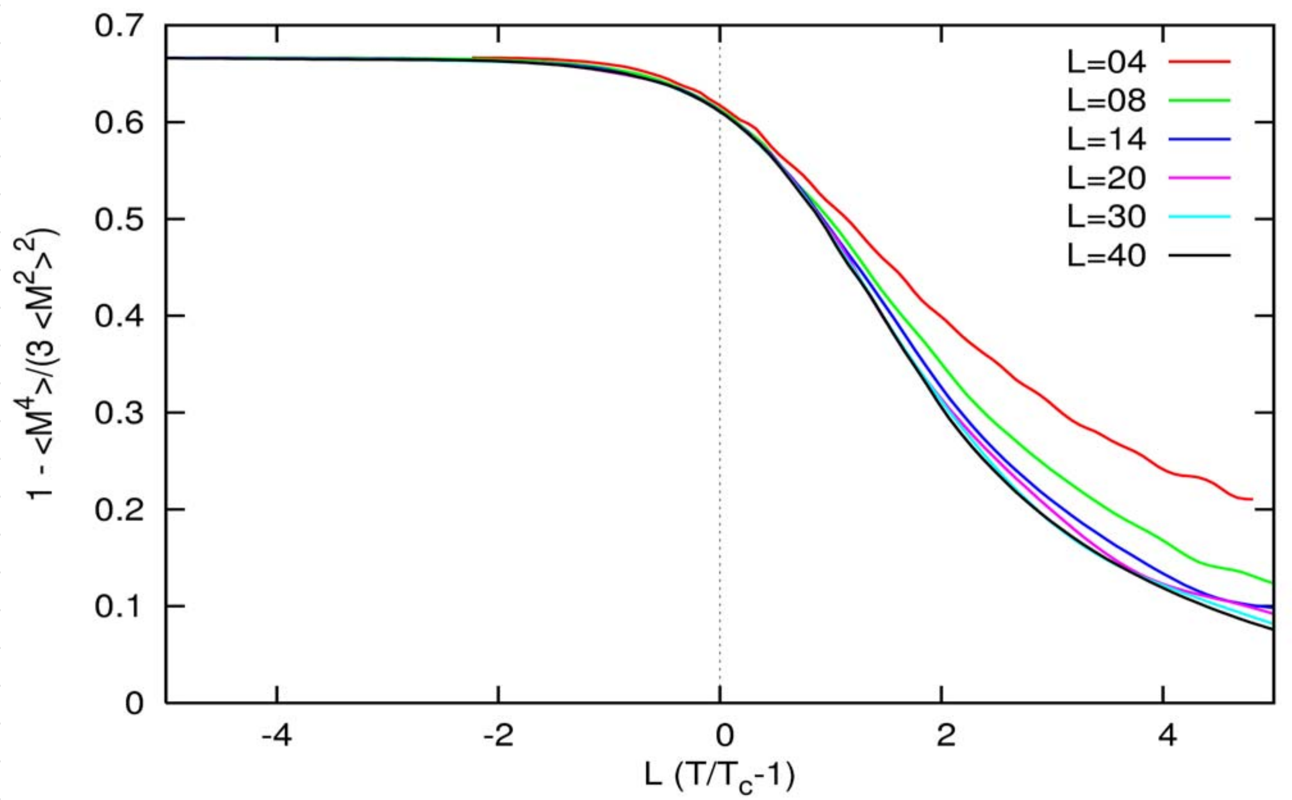
Due to the absence of a renormalizing prefactor, curves $U_4(T)$ obtained for different L intersect asymptotically at $T = T_c$ (but are expanded horizontally by different factors $L^{1/\nu}$).

Example: MC simulation data for 2d square lattice Ising model



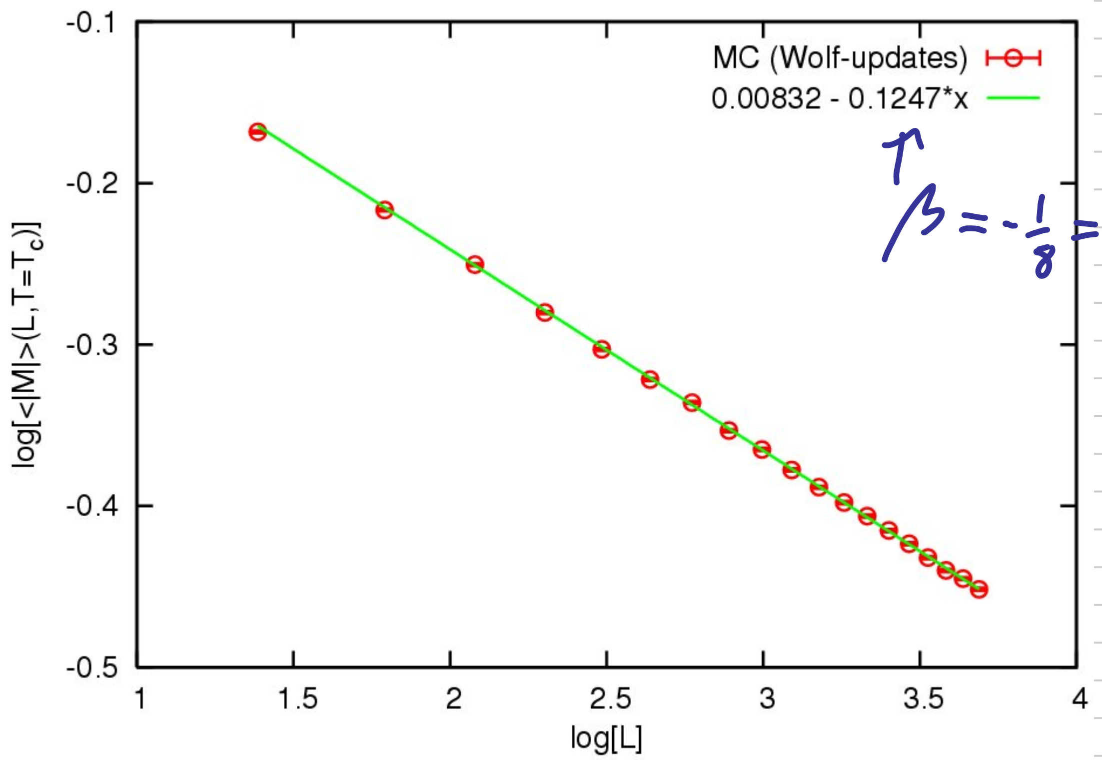
One observes: $U_4 \xrightarrow{L\to\infty} \begin{cases} \frac{2}{3} & \text{for } T<T_c \\ U^* \approx 0.61 & \text{" } T=T_c \\ 0 & \text{" } T>T_c \end{cases}$

Scaling plot (Wolff algorithm, smoothed curves)



Once T_c is known, critical exponents can be determined, e.g., using logarithmic plots.

Example: determination of exponent β from magnetization



$\beta = -\frac{1}{8} = -0.125$