Hier Ferromagnetisches Modell (Vorlesung: AF)

Beispiel: Heisenberg-Modell, $N=3$ (periodisch):

$m_z = 3$: $|3,1\rangle = \uparrow\uparrow\uparrow$; $H|3,1\rangle = -3J|3,1\rangle$

$m_z = 1$: Zustände $\downarrow\uparrow\uparrow$, $\uparrow\downarrow\uparrow$, $\uparrow\uparrow\downarrow$

antisymm Kombination muss EV sein

symm

muss EV sein

$q=0$	$\frac{1}{\sqrt{3}}$	1	1	1	$|1,1\rangle$
$q=\frac{2\pi}{3}$	$\sqrt{\frac{2}{3}}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$|1,2\rangle$
	$\frac{1}{\sqrt{2}}$	0	1	-1	$|1,3\rangle$

$$H \hat{=} -J\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$$

$$H|1,1\rangle = -\frac{J}{\sqrt{3}}\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = -3J|1,1\rangle$$

$$H|1,2\rangle = -\sqrt{\tfrac{2}{3}}\,J\begin{pmatrix} -3 \\ \frac{3}{2} \\ \frac{3}{2} \end{pmatrix} = -3J|1,2\rangle$$

$$H|1,3\rangle = -\frac{J}{\sqrt{2}}\begin{pmatrix} 0 \\ -3 \\ 3 \end{pmatrix} = 3J|1,3\rangle$$

$\rightsquigarrow$ EW $(m_z=1)$: $-3J$, $-3J$, $3J$

Beispiel: Heisenberg-Modell, $N=4$

Hamilton-Matrix für $m_z=0$:

$$\hat{H} = -J \begin{array}{c} \\ \\ \end{array} \begin{array}{cc} \begin{array}{cccccc} \uparrow\downarrow\uparrow\downarrow & \downarrow\uparrow\downarrow\uparrow & \uparrow\uparrow\downarrow\downarrow & \downarrow\uparrow\uparrow\downarrow & \downarrow\downarrow\uparrow\uparrow & \uparrow\downarrow\uparrow\downarrow \end{array} & \\ \begin{pmatrix} -4 & 0 & 2 & 2 & 2 & 2 \\ 0 & -4 & 2 & 2 & 2 & 2 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{array}{c} \uparrow\downarrow\uparrow\downarrow \\ \downarrow\uparrow\downarrow\uparrow \\ \uparrow\uparrow\downarrow\downarrow \\ \downarrow\uparrow\uparrow\downarrow \\ \downarrow\downarrow\uparrow\uparrow \\ \uparrow\downarrow\downarrow\uparrow \end{array} \end{array}$$

$m_z = 0$:

(i) alternierend

		$\uparrow\downarrow\uparrow\downarrow$	$\downarrow\uparrow\downarrow\uparrow$	
$p=0$:	$\frac{1}{\sqrt{2}}$	$(1$	$1)$	$\lvert 0,1\rangle$
$p=\pi$:	$\frac{1}{\sqrt{2}}$	$(1$	$-1)$	$\lvert 0,2\rangle$

mögliche Kopplungen

(ii) phasenseparient

		$\uparrow\uparrow\downarrow\downarrow$	$\downarrow\uparrow\uparrow\downarrow$	$\downarrow\downarrow\uparrow\uparrow$	$\uparrow\downarrow\downarrow\uparrow$	
$p=0$	$\frac{1}{2}$	1	1	1	1	$\lvert 0,3\rangle$
$p=\frac{\pi}{2}$	$\frac{1}{\sqrt{2}}$	1	0	-1	0	$\lvert 0,4\rangle$
		0	1	0	-1	$\lvert 0,5\rangle$
$p=\pi$	$\frac{1}{2}$	1	-1	1	-1	$\lvert 0,6\rangle$

Impuls $p=0$:

$$\Rightarrow \quad H\,|0,1\rangle = -J\,\frac{1}{\sqrt{2}}\begin{pmatrix}-4\\-4\\4\\4\\4\\4\end{pmatrix} = 4J\,|0,1\rangle - 4\sqrt{2}\,J\,|0,3\rangle$$

$$H\,|0,3\rangle = -\frac{J}{2}\begin{pmatrix}8\\8\\0\\0\\0\\0\end{pmatrix} = -4\sqrt{2}\,J\,|0,1\rangle$$

$$\Rightarrow \quad H \mathrel{\hat{=}} J \begin{array}{c c} \begin{array}{cc} |0,1\rangle & |0,3\rangle \end{array} & \\ \begin{pmatrix} 4 & -4\sqrt{2} \\ -4\sqrt{2} & 0 \end{pmatrix} & \begin{array}{c} |0,1\rangle \\ |0,3\rangle \end{array} \end{array}$$

$$p(\lambda) = (\lambda-4)\lambda - 32 = \lambda^2 - 4\lambda - 32$$

$$\lambda_\pm = 2 \pm \sqrt{36} = 2 \pm 6 = \begin{cases} 8 \\ -4 \end{cases}$$

EW 8: $\begin{pmatrix} 4 & 4\sqrt{2} \\ 4\sqrt{2} & 8 \end{pmatrix}\vec{x} = 0 \Rightarrow \vec{x} = \frac{1}{\sqrt{3}}\begin{pmatrix}\sqrt{2}\\-1\end{pmatrix}$

$\Rightarrow \quad \sqrt{\frac{2}{3}}\,|0,1\rangle - \frac{1}{\sqrt{3}}\,|0,3\rangle$ ist EV zum EW $8J$.

EW -4: $\begin{pmatrix} -8 & 4\sqrt{2} \\ 4\sqrt{2} & -4 \end{pmatrix}\vec{x} = 0 \Rightarrow \vec{x} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\\sqrt{2}\end{pmatrix}$

$\Rightarrow \quad \frac{1}{\sqrt{3}}\,|0,1\rangle + \sqrt{\frac{2}{3}}\,|0,3\rangle$ ist EV zum EW $-4J$

$p=\pi$:

$$H\,|0,2\rangle = -\frac{J}{\sqrt{2}}\begin{pmatrix}-4\\4\\0\\0\\0\\0\end{pmatrix} = 4J\,|0,2\rangle \quad \text{EV}$$

$$H\,|0,6\rangle = -\frac{J}{2}\begin{pmatrix}8\\0\\0\\0\\0\\0\end{pmatrix} = 0J\,|0,6\rangle \quad \text{EV}$$

$p=\frac{\pi}{2}, \frac{3\pi}{2}$: $\quad H\,|0,4\rangle = -\frac{J}{\sqrt{2}}\begin{pmatrix}0\\0\\0\\0\\0\\0\end{pmatrix} = 0J\,|0,4\rangle \quad$ EV

$$H|0,5\rangle = -\frac{J}{\sqrt{2}}\begin{pmatrix}0\\0\\0\\0\\0\\0\end{pmatrix} = 0J|0,5\rangle \quad \text{EV}$$

$$\Rightarrow \text{EVs:}\quad \underset{\substack{p=0\\E=8J}}{\frac{1}{2\sqrt{3}}\begin{pmatrix}2\\2\\-1\\-1\\-1\\-1\end{pmatrix}},\ \underset{\substack{p=0\\E=-4J}}{\frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix}},\ \underset{\substack{p=\pi\\E=4J}}{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\\0\\0\\0\end{pmatrix}},\ \underset{\substack{p=\pi\\E=0J}}{\frac{1}{2}\begin{pmatrix}0\\0\\1\\-1\\1\\-1\end{pmatrix}},\ \underset{\substack{p=\frac{\pi}{2},\frac{3\pi}{2}\\E=0J}}{\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\0\\-1\\0\end{pmatrix}},\ \underset{E=0J}{\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\0\\1\\0\\-1\end{pmatrix}}$$

$m_z = 4$: $\quad p=0 \quad \uparrow\uparrow\uparrow\uparrow = |4,1\rangle;\quad H|4,1\rangle = -4|4,1\rangle$

$m_z = 2$:

		$\downarrow\uparrow\uparrow\uparrow$	$\uparrow\downarrow\uparrow\uparrow$	$\uparrow\uparrow\downarrow\uparrow$	$\uparrow\uparrow\uparrow\downarrow$	
	$p=0$: $\frac{1}{2}$	(1	1	1	1)	$\|2,1\rangle$
cos	$p=\frac{\pi}{2}$: $\frac{1}{\sqrt{2}}$	(1	0	-1	0)	$\|2,2\rangle$
sin	$1/\sqrt{2}$	(0	1	0	-1)	$\|2,3\rangle$
	$p=\pi$: $\frac{1}{2}$	(1	-1	1	-1)	$\|2,4\rangle$

$p=0$ $$H|2,1\rangle = -\frac{J}{2}\begin{pmatrix}0&2&0&2\\2&0&2&0\\0&2&0&2\\2&0&2&0\end{pmatrix}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = -\frac{J}{2}\begin{pmatrix}4\\4\\4\\4\end{pmatrix} = -4J|2,1\rangle$$

$p=\frac{\pi}{2},\frac{3\pi}{2}$ $$H|2,2\rangle = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} = 0J|2,2\rangle \qquad H|2,3\rangle = 0J|2,3\rangle$$

$p=\pi$ $$H|2,4\rangle = -\frac{J}{2}\begin{pmatrix}-4\\4\\-4\\4\end{pmatrix} = 4J|2,4\rangle$$

Damit haben wir das 4Spin-Heisenberg-Modell mit periodischen Randbedingungen vollständig analytisch gelöst – durch Ausnutzen der Symmetrien!

Für offene Randbed. siehe:

http://komet337.physik.uni-mainz.de/Bluemer/Scripts/num-meth-ss07-v10.pdf