itinerant ferromagnetism (Hubbard, 1963; Gutzwiller, 1963; Kanamori, 1963), have become a major subfield of condensed matter theory. Since the hopping and interaction terms in (1.14) do not commute, they cannot be simultaneously diagonalized which leaves the treatment of the Hubbard model highly nontrivial. In the (grand) canonical ensemble, the model is fully specified by choosing t and U plus the underlying lattice, the temperature T, and the average band filling $n := \frac{1}{L}\sum_{i=1}^{L}\langle \hat{n}_{i\uparrow} + \hat{n}_{i\downarrow}\rangle$ (or, equivalently, the chemical potential μ).

For dimension[9] $d = 1$, many properties of the Hubbard model such as the ground state wave function, magnetic susceptibility, Drude weight, and excitation spectra can be exactly calculated using the Bethe ansatz (Bethe, 1931; Lieb and Wu, 1968) in the thermodynamic limit. For higher dimensions, little is known rigorously. The generic low-temperature phase for half-filled ($n = 1$) bipartite lattices (for a definition, see Sec. 2.1) and $d \geq 2$ (only at $T = 0$ for $d = 2$) is antiferromagnetic (AF), i.e., both sublattices have a finite (opposite) magnetization. This phase is found in weak coupling ($U \to 0$; qualitatively in Hartree-Fock, i.e., Slater mean-field theory) and strong coupling (Anderson's "super-exchange" mechanism) for $d > 2$ and can also be approached by renormalization group methods, at least for $d = 2$ (Halboth and Metzner, 2000; Honerkamp and Salmhofer, 2001). Ferromagnetism is rigorously established in the pure Hubbard model only in the thermodynamically irrelevant case of half filling minus one electron at $U = \infty$ for a variety of lattice types (in dimensions $d \geq 2$; in one dimension it is excluded by the Lieb-Mattis theorem) and also for so-called "flat band systems" (Mielke and Tasaki, 1993). For $n = 1$ and $U \to \infty$, the Hubbard model can be mapped to the spin-1/2 Heisenberg model for which a vast literature exists; less is known about the t-J model to which the Hubbard model is closely related for $U \to \infty$ and $n \neq 1$. In $d = 2$, the Hubbard model is widely used for modeling high-T_c superconductors. While the existence of long-range order in the d-wave pairing is rigorously excluded (Mermin and Wagner, 1966; Su and Suzuki, 1998) at finite temperatures for narrow-band models, instabilities towards such order can still be inferred from corresponding correlation functions or from calculations at $T = 0$. For a more comprehensive overview over the physics of the Hubbard model see, e.g., the proceedings edited by Baeriswyl, Campbell, Carmelo, Guinea, and Louis (1995).

The one-band Hubbard model represents a highly idealized view on strongly correlated materials. Its applicability to d or f electron systems is a priori questionable since the partially filled bands correspond to atomic orbitals which are 5-fold and 7-fold degenerate (for each spin direction), respectively. While this degeneracy is at least partially lifted by crystal field effects, a more realistic description can potentially be achieved using multi-band versions of the Hubbard model, possibly also

[9] Note that for a lattice model with local Coulomb interaction, the dimensionality of the problem is completely determined by the hopping matrix elements t_{ij} in (1.9). The physics of the problem is of lower dimension d than the host lattice if the sublattice formed by all sites connected to one given site via t_{ij} is topologically equivalent to a *regular* lattice of dimension d since then the Hamiltonian factorizes into identical contributions from each of the unconnected sublattices. This fact explains why even some (unisotropic) bulk systems in our three-dimensional world may be modeled as one-dimensional or two-dimensional. Deviations from this idealized behavior arise both from residual hopping matrix elements and from long-range Coulomb interactions.