Thus, the operators $\hat{\boldsymbol{h}}_i$ correspond to conserved quantities and may be replaced by their expectation values. Choosing the z axis as the quantization axis, this procedure is equivalent to an application of the Hartree decoupling scheme $\hat{\boldsymbol{S}}_i \cdot \hat{\boldsymbol{S}}_j \longrightarrow \hat{S}_i^z \hat{S}_j^z \longrightarrow \langle \hat{S}_i^z \rangle \hat{S}_j^z + \hat{S}_i^z \langle \hat{S}_j^z \rangle - \langle \hat{S}_i^z \rangle \langle \hat{S}_j^z \rangle$ to (1.15) which leads to the mean-field Hamiltonian

$$\hat{H}_{\mathrm{MF}} = -2J^* \sum_i \hat{S}_i^z h_i + J^* \sum_i \langle \hat{S}_i^z \rangle h_i \ . \tag{1.17}$$

The same form is obtained in the $Z \to \infty$ limit of the Ising Hamiltonian $\hat{H}_{\mathrm{Ising}} = -J \sum_{\langle ij \rangle} \hat{S}_i^z \hat{S}_j^z$, the anisotropic limit of the Heisenberg model.

On the mean-field level, the thermodynamics of the ferromagnetic ($J > 0$) and antiferromagnetic ($J < 0$) models are the same. However, for the quantum Heisenberg model in $d = 3$, ferromagnetic and antiferromagnetic order differ substantially in character since the fully polarized antiferromagnet is not an eigenstate. The mean-field critical temperature of $T_{\mathrm{c}}^{\mathrm{MF}}/J^* = 0.5$ can be compared to the numerically exact Curie temperature of the ferromagnetic 3-d Heisenberg model where $T_{\mathrm{c}}/J^* = 0.28$, 0.325, and 0.346 for simple cubic (sc), body centered cubic (bcc), and face centered cubic (fcc) lattices, respectively (Kittel, 1996). For the Ising model, a thermodynamically consistent approximation which also captures $\mathcal{O}(1/Z)$ corrections to the mean-field theory is given by the so-called spherical model (Berlin and Kac, 1952) which also applies to the Heisenberg model for $T > T_{\mathrm{c}}$ (Brout, 1960; Brout, 1961).[12] An application of this $\mathcal{O}(1/Z)$ theory to the 3-d ferromagnetic Heisenberg model yields $T_{\mathrm{c}}/J^* = 0.37$, 0.36, and 0.33 for sc, bcc, and fcc lattices, respectively (Brout, 1960). Here, $1/Z$ corrections already greatly improve the mean-field estimates for $Z \sim 10$.

1.2.2 Limit $Z \to \infty$ for Fermions

In order to define a nontrivial $Z \to \infty$ limit of the Hubbard model (1.14) one has to scale its parameters so that the competition between kinetic and potential energy is retained. Since U parameterizes the local interaction, it is independent of the coordination number. The scaling (Metzner and Vollhardt, 1989)

$$t = t^*/\sqrt{Z} \tag{1.18}$$

of the nearest neighbor (NN) hopping amplitudes ensures constant, finite variance of its energy eigenvalue spectrum,

$$< \epsilon^2 >= t^{*2} \tag{1.19}$$

as derived for arbitrary (pseudo-) lattice type, dimension, and hopping range in subsection 2.2.2. A convenient class of lattices with arbitrarily high (even) coordination number is formed by the hypercubic (hc) lattices which for dimension d (with $Z = 2d$)

[12] Among the pathologies encountered in earlier $1/Z$ expansions for spin models is a violation of the sum rule $\sum_{i=1}^{N} (S_i^z)^2 = N/4$. An enforcement of this equality using a Lagrange parameter leads to a reselection of diagrams of order $1/Z^2$. Only then one observes, e.g., a divergence of the susceptibility for $T \to T_{\mathrm{c}}^+$ as required for a 2$^{\mathrm{nd}}$ order phase transition (Brout, 1961).