and unit lattice spacing have the dispersion

$$\epsilon_{\boldsymbol{k}} = -2t \sum_{l=1}^{d} \cos(k_l) \,. \tag{1.20}$$

In the limit $d \to \infty$ and for generic $\boldsymbol{k}$, this expression turns into an infinite sum over statistically independent contributions to which the central limit theorem applies. Consequently, the noninteracting density of states (DOS) acquires a Gaussian form,

$$\rho^{\mathrm{hc}}(\epsilon) = \frac{1}{\sqrt{2\pi}\, t^*} \exp\left(-\frac{\epsilon^2}{2t^{*2}}\right) \,. \tag{1.21}$$

One might be disturbed by the observation that this DOS is unbounded both at low and high energies which makes the low-density limit $n \to 0$ unphysical since the kinetic energy per electron diverges. This pathology is easily linked to the additive independent energy contributions from each dimension to the total dispersion for the hc lattice.[13] In fact, an infinite number (of measure 0) of so-called nongeneric momenta $\boldsymbol{k}$ with $\epsilon_{\boldsymbol{k}} = \pm\infty$ exists such as the center $\mathbf{0} := (0, 0, 0, \dots)$ of the Brillouin zone or the antiferromagnetic wave vector $\boldsymbol{Q} := (\pi, \pi, \pi, \dots)$ (for lattice spacing $a = 1$). For finite electron density $n$, however, all states with $\epsilon_{\boldsymbol{k}} = -\infty$ ($\epsilon_{\boldsymbol{k}} = \infty$) remain full (empty) at all finite temperatures and values of the interaction $U$ and thus essentially drop out of the problem. Also, the energy per lattice site remains finite for arbitrary filling. An observation related to the appearance of infinite energy contributions is that, for $d \to \infty$, the 2-particle density of states with momentum transfer $\boldsymbol{q}$

$$D_{\boldsymbol{q}}(\epsilon_1, \epsilon_2) := \frac{1}{L} \sum_{\boldsymbol{k}} \delta(\epsilon_1 - \epsilon_{\boldsymbol{k}+\boldsymbol{q}})\delta(\epsilon_2 - \epsilon_{\boldsymbol{k}}) \tag{1.22}$$

only depends on $\boldsymbol{q}$ via a scalar $\eta$ and reads (for hc DOS)

$$D_{\boldsymbol{q}}(\epsilon_1, \epsilon_2) = \frac{\exp\left(-\frac{(\epsilon_1+\epsilon_2)^2}{4t^{*2}(1+\eta_{\boldsymbol{q}})}\right)}{t^*\sqrt{2\pi(1+\eta_{\boldsymbol{q}})}} \; \frac{\exp\left(-\frac{(\epsilon_1-\epsilon_2)^2}{4t^{*2}(1-\eta_{\boldsymbol{q}})}\right)}{t^*\sqrt{2\pi(1-\eta_{\boldsymbol{q}})}} \,, \tag{1.23}$$

which implicitly defines $\eta_{\boldsymbol{q}}$. Specifically, $\eta_{\boldsymbol{q}} = \epsilon_{\boldsymbol{q}}/(t^*\sqrt{2d})$ for the hc lattice. Evidently, $\eta_{\boldsymbol{q}}$ vanishes generically (i.e., for all momenta with finite energy) and thus $D_{\boldsymbol{q}}$ factorizes for all generic $\boldsymbol{q}$ (Müller-Hartmann, 1989*a*; Müller-Hartmann, 1989*b*; van Dongen, Gebhard, and Vollhardt, 1989)

$$D_{\boldsymbol{q}}(\epsilon_1, \epsilon_2) = \rho(\epsilon_1)\, \rho(\epsilon_2) \quad \text{for generic } \boldsymbol{q}. \tag{1.24}$$

Physically, it is important that some correlations remain in $D_{\boldsymbol{q}}(\epsilon_1, \epsilon_2)$ even in the limit $Z = \infty$. There is necessarily always full correlation for $\boldsymbol{q} = \mathbf{0}$: $\eta(\mathbf{0}) = 1$. Furthermore, perfect nesting implies $\eta(\boldsymbol{Q}) = -1$ for any bipartite lattice.[14]

[13] We will explore alternative lattice types in chapter 2 and find that sharp band edges can be obtained in the limit $Z \to \infty$ not only for the Bethe pseudo-lattice, but also for regular lattices (with full momentum space). For the rest of this section, however, we focus on the more conventional hc lattice.

[14] Note that the presence of extended hopping along the axes can destroy perfect nesting (Müller-Hartmann, 1989*a*) without changing the DOS in $d = \infty$; see subsection 2.1.3.