where the matrix $\boldsymbol{\mathcal{G}}_\sigma$ consists of elements $\mathcal{G}_{\sigma ll'} \equiv \mathcal{G}_\sigma(l\Delta\tau - l'\Delta\tau)$, we apply (1.37) and obtain to lowest order

$$\begin{aligned}\exp\left(\mathcal{A}_\Lambda[\psi,\psi^*,\mathcal{G},U]\right) &= \prod_{l=0}^{\Lambda-1}\Bigg[\exp\Big((\Delta\tau)^2\sum_\sigma\sum_{l'=0}^{\Lambda-1}\psi^*_{\sigma l}(\boldsymbol{\mathcal{G}}_\sigma^{-1})_{ll'}\,\psi_{\sigma l'}\Big)\\ &\qquad\times\ \exp\left(-\Delta\tau\, U\,\psi^*_{\uparrow l}\,\psi_{\uparrow l}\,\psi^*_{\downarrow l}\,\psi_{\downarrow l}\right)\Bigg].\end{aligned} \tag{1.39}$$

Shifting the chemical potential by $U/2$, the four-fermion term can be rewritten as a square of two-fermion terms, which makes it suitable for the following discrete Hubbard-Stratonovich transformation (Hirsch, 1983):

$$\exp\left(\frac{\Delta\tau U}{2}(\psi^*_{\uparrow l}\,\psi_{\uparrow l} - \psi^*_{\downarrow l}\,\psi_{\downarrow l})^2\right) = \frac{1}{2}\sum_{s_l=\pm 1}\exp\left(\lambda s_l(\psi^*_{\uparrow l}\,\psi_{\uparrow l} - \psi^*_{\downarrow l}\,\psi_{\downarrow l})\right) \tag{1.40}$$

with $\cosh\lambda = \exp(\Delta\tau U/2)$. Here, the interaction between electrons is replaced by the interaction with an auxiliary binary field $\{s\}$ with components s_l for $0 \le l \le \Lambda$. Acting like a local, but time-dependent magnetic field, $\{s\}$ can be regarded as an ensemble of Ising spins.

These transformations yield an expression for the functional integral

$$G_{\sigma l_1 l_2} = \frac{1}{\mathcal{Z}}\sum_{\{s\}}\int\mathcal{D}[\psi]\,\mathcal{D}[\psi^*]\ \psi^*_{\sigma l_1}\psi_{\sigma l_2}\exp\Big(\sum_{\sigma,l,l'}\psi^*_{\sigma l}M^{s_l}_{\sigma ll'}\psi_{\sigma l'}\Big), \tag{1.41}$$

with[24]

$$M^{s_l}_{\sigma ll'} = (\Delta\tau)^2(\boldsymbol{\mathcal{G}}_\sigma^{-1})_{ll'} - \lambda\sigma\delta_{ll'}s_l, \tag{1.42}$$

where in (1.41) the sum is taken over all configurations of the Ising spin field, and each term of the sum involves independent fermions only. Now Wick's theorem (see, e.g., Negele and Orland, 1987) can be applied to get the solution

$$G_{\sigma ll'} = \frac{1}{\mathcal{Z}}\sum_{\{s\}}\big(\boldsymbol{M}_\sigma^{\{s\}}\big)^{-1}_{ll'}\det\boldsymbol{M}_\uparrow^{\{s\}}\det\boldsymbol{M}_\downarrow^{\{s\}}, \tag{1.43}$$

where $\boldsymbol{M}_\sigma^{\{s\}}$ is the matrix with elements $M^{s_l}_{\sigma ll'}$, and the partition function has the value

$$\mathcal{Z} = \sum_{\{s\}}\det\boldsymbol{M}_\uparrow^{\{s\}}\det\boldsymbol{M}_\downarrow^{\{s\}}. \tag{1.44}$$

Computing one of the 2^Λ terms in (1.43) directly from definition (1.42) is an operation of order $\mathcal{O}(\Lambda^3)$. If the terms are ordered in a way so that successive configurations $\{s\}$ and $\{s\}'$ only differ by one flipped spin $s_l \to -s_l$ then all matrices and determinants can be updated at a cost of $\mathcal{O}(\Lambda^2)$ (Blankenbecler, Scalapino, and Sugar, 1981). Only for $\Lambda \lesssim 24$ can all terms be summed up exactly. Computations at larger Λ are made possible by Monte Carlo importance sampling which reduces the number of terms that have to be calculated explicitly from 2^Λ to order $\mathcal{O}(\Lambda)$.

[24] A more precise form including subleading corrections is $M^{s_l}_{\sigma ll'} = (\Delta\tau)^2\,(\boldsymbol{\mathcal{G}}_\sigma^{-1})_{ll'}\,\mathrm{e}^{\lambda\sigma s_{l'}} + \delta_{ll'}\,\big(1 - \mathrm{e}^{\lambda\sigma s_l}\big)$ (Held, 1999; Georges et al., 1996).