fulfill a recursion relation in $d = \infty$,

$$
\begin{aligned}
B_{D+1}(\boldsymbol{k}) &= B_1(\boldsymbol{k})\, B_D(\boldsymbol{k}) - D\, B_{D-1}(\boldsymbol{k})\, \frac{2}{d} \sum_{\alpha=1}^{d} \cos^2(k_\alpha) + \mathcal{O}(1/d) \\
&= B_1(\boldsymbol{k})\, B_D(\boldsymbol{k}) - D\, B_{D-1}(\boldsymbol{k}) + \mathcal{O}(1/\sqrt{d})\,.
\end{aligned}
\tag{2.49}
$$

To leading order, this relation is very similar to that of the Hermite polynomials[23]

$$
\mathrm{He}_{n+1}(x) = x\, \mathrm{He}_n(x) - n\, \mathrm{He}_{n-1}(x)\,. \tag{2.50}
$$

From (2.49), (2.50), and $B_0 = 1$ we conclude that $B_D(\boldsymbol{k}) = \mathrm{He}_D(B_1(\boldsymbol{k}))$. Furthermore, we can express the argument x in (2.50) as the dispersion of the hypercubic lattice with a positive hopping term [i.e., $t^* = -1$ in (1.14)] and unit variance: $B_1(\boldsymbol{k}) = \epsilon^{\mathrm{hc}}_{\boldsymbol{k}}$. Consequently, we obtain

$$
\epsilon_D(\boldsymbol{k}) = t_D \frac{2^D}{D!} \left(\frac{d}{2}\right)^{D/2} \mathrm{He}_D(B_1(\boldsymbol{k})) \tag{2.51}
$$

and

$$
\boxed{\epsilon(\boldsymbol{k}) = \sum_{D=1}^{\infty} \frac{t^*_D}{\sqrt{D!}}\, \mathrm{He}_D(\epsilon^{\mathrm{hc}}_{\boldsymbol{k}}) \quad =: \quad \mathcal{F}(\epsilon^{\mathrm{hc}}_{\boldsymbol{k}})\,.} \tag{2.52}
$$

Finding not only an effectively one-dimensional expression, but even a closed form (2.52) for the general dispersion of a hypercubic lattice with arbitrary-range hopping in terms of scaled hopping matrix elements is clearly a significant result. On the one hand, this expression applies to all such models where the hopping amplitudes depend on taxi-cab distance and therefore systematically classifies most models studied in the DMFT literature. On the other hand, (2.52) will enable us to explore information hidden in any given DOS, namely the dispersion and the scaled hopping matrix elements of a corresponding model (in $d = \infty$), as we will show in the following. Since any analytic function can be expanded in Hermite polynomials using the orthogonality relation

$$
\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{He}_n(x)\, \mathrm{He}_m(x)\, \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}} = n!\, \delta_{mn}, \tag{2.53}
$$

we can find hopping amplitudes t^*_D so that $\epsilon(\boldsymbol{k}) = \mathcal{F}(\epsilon^{\mathrm{hc}}_{\boldsymbol{k}})$ for an arbitrary analytic function $\mathcal{F}$. As an illustration of this expansion, we rescale the Hermite polynomials in Fig. 2.14 a to

$$
p_D(x) := \mathrm{He}_D(x)/\sqrt{D!}\,. \tag{2.54}
$$

The full transformation function is then a linear combination of the basis functions p_D where the Euclidean norm of the vector of expansion coefficients t^*_D sets the energy scale for the resulting DOS.

[23]For the two different definitions of the Hermite polynomials with $H_n(x) = 2^{n/2}\mathrm{He}_n(\sqrt{2}x)$ see, e.g., Abramowitz and Stegun (1972).