fully determined by the first derivative of the transformation function at the origin,

$$\frac{1}{\rho(\epsilon=0)} = \sqrt{2\pi}\,\mathcal{F}'(0) = \sqrt{2\pi}\sum_{n=0}^{\infty} t^*_{2n+1}(-1)^n \frac{(2n+1)!!}{\sqrt{(2n+1)!}}\,. \tag{2.71}$$

This series is quickly converging for the tight-binding model corresponding to a Bethe DOS as seen in Fig. 2.18 and Fig. 2.19.[29]

### 2.4.3 Finite Dimensionality

After having verified that the properties of the tight-binding model with semi-elliptic DOS derived in this section are robust with respect to truncation, we can now study the robustness with respect to application in finite dimensions and thereby distinguish generic from more artificial features. For this purpose, we explicitly calculate the dispersion for the redefined Bethe model [i.e., the model defined by (2.68)], truncated at 3$^{\text{rd}}$ or 5$^{\text{th}}$ order hopping. Here, we evaluate $B_D(\boldsymbol{k})$ as defined in (2.48) without resorting to any large-$d$ approximations for the involved sums or the number of equivalent hopping terms; in particular, the approximation in (2.44) is not used.[30] In the necessary summations, the exclusion of terms with equal indices poses a combinatorial problem which can be solved iteratively or as a linear system of equations. In order to illustrate the basic considerations, we show how the summation restrictions are replaced for $D = 2, 3$: $\sum_{i\neq j} = \sum_{ij}(1-\delta_{ij})$; $\sum_{i\neq j\neq k} = \sum_{ijk}(1-\delta_{ij}-\delta_{ik}-\delta_{jk}+2\delta_{ijk})$, where the generalized $\delta$-function equals 1 if all indices are equal and is 0 otherwise. For the sake of brevity, we will in the following state the results for $A_D(\boldsymbol{k}) := (2/d)^{-d/2}B_D(\boldsymbol{k})$, from which the exact $d$-dimensional dispersion can be obtained in the form

$$\epsilon_D(\boldsymbol{k}) = t^*_D \frac{2^{D/2}}{D!\sqrt{\binom{d}{D}}} A_D(\boldsymbol{k}) \tag{2.72}$$

and which agrees with (2.47) in the limit $d \gg D$. The coefficient functions read

$$A_1(\boldsymbol{k}) = \sum_{\alpha=1}^{d}\cos(k_\alpha); \qquad A_2(\boldsymbol{k}) = \Big(\sum_\alpha \cos(k_\alpha)\Big)^2 - \sum_\alpha \cos^2(k_\alpha)$$

[29] Note that a gapped DOS as in the case of the flat band DOS considered above can only arise when the sum in (2.71) diverges which means that the hopping matrix elements $t^*_n$ decay more slowly than $n^{-5/4}$.

[30] We keep, however, the selection of hopping terms defined at the beginning of this section, i.e., do not consider extended hopping along the cartesic axes.