The Kramers-Kronig equation for the real part of $\epsilon(\omega)$ reads

$$\begin{aligned}
\operatorname{Re}\epsilon(\omega) &= 1+\frac{1}{\pi}P\int\limits_{-\infty}^{\infty}\mathrm{d}\omega'\,\operatorname{Im}\epsilon(\omega')\frac{1}{\omega'-\omega} && (4.15)\\
&= 1+\frac{2}{\pi}\int\limits_{0}^{\infty}\mathrm{d}\omega'\,\operatorname{Im}\epsilon(\omega')\frac{\omega'}{\omega'^2-\omega^2}, && (4.16)
\end{aligned}$$

where in the second line we used the fact that $\epsilon(-\omega)=\epsilon^*(\omega)$ since $\boldsymbol{E}$ and $\boldsymbol{D}$ are real. Under the assumption that excitations (at $\boldsymbol{q}=\boldsymbol{0}$) are restricted to a finite energy range $0<\omega<\omega_{\max}$, we can take the $\omega\to\infty$ limit of (4.16),

$$\begin{aligned}
\lim_{\omega\to\infty}\omega^2\big(1-\operatorname{Re}\epsilon(\omega)\big) &= \frac{2}{\pi}\lim_{\omega\to\infty}\int\limits_{o}^{\omega_{\max}}\mathrm{d}\omega'\,\operatorname{Im}\epsilon(\omega')\,\omega'\left(1+\frac{\omega'^2}{\omega^2}-\ldots\right)\\
&= \frac{2}{\pi}\int\limits_{0}^{\infty}\mathrm{d}\omega'\,\operatorname{Im}\epsilon(\omega')\omega' \;=:\; \omega_{\mathrm{p}}^2, && (4.17)
\end{aligned}$$

where ω_{p} is the plasma frequency. Using (4.17) and (4.7) we arrive at the f-sum rule for the optical conductivity,

$$\int\limits_{0}^{\infty}\mathrm{d}\omega\,\operatorname{Re}\sigma(\omega)=\frac{\omega_{\mathrm{p}}^2}{8}\,. \tag{4.18}$$

Other useful f-sum rules which follow from (4.17) are

$$\begin{aligned}
\int\limits_{0}^{\infty}\mathrm{d}\omega\,\omega k(\omega) &= \frac{\pi}{4}\omega_{\mathrm{p}}^2 && (4.19)\\
\int\limits_{0}^{\infty}\mathrm{d}\omega\,\omega\operatorname{Im}\frac{-1}{\epsilon(\omega)} &= \frac{\pi}{2}\omega_{\mathrm{p}}^2\,. && (4.20)
\end{aligned}$$

Note that the limit $\omega\to\infty$ used in the above equations is problematic for several (related) reasons: most importantly, photons of high frequency have a short wavelength so that any theory which depends on the long-wavelength limit $\boldsymbol{q}\to 0$ breaks down. While the range of validity could be enlarged by considering finite q, ultimately the homogeneity condition is violated when $\omega\gtrsim\frac{c}{a}$, for lattice spacing a. In fact, the response at high frequencies, i.e., (inelastic) x-ray diffraction, must show strong angular dependence due to the lattice structure factor. Furthermore, the excitation of core electrons will lead to relativistic effects for systems containing heavy nuclei.

Theoretically, these problems may be avoided by considering the jellium model of interacting electrons in a neutralizing homogeneous background. In this case, the