where n_0 is the average density. The remaining problem is the calculation of $\langle \boldsymbol{j}(\boldsymbol{r},t)\rangle_A$ in linear order in $\boldsymbol{A}$. For this purpose, it suffices to consider the linear perturbing part of the Hamiltonian[12] which may be expressed as

$$H'(t) = -\frac{1}{c}\int \mathrm{d}\boldsymbol{r}\, \boldsymbol{j}(\boldsymbol{r},t)\cdot \boldsymbol{A}(\boldsymbol{r},t)\,. \tag{4.31}$$

From (D.2) we have for periodic field $\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}_{\boldsymbol{q}}\mathrm{e}^{i(\boldsymbol{q}\cdot\boldsymbol{r}-\omega t)}$

$$\boldsymbol{A}(\boldsymbol{r},t) = -\frac{ic}{\omega}\boldsymbol{E}(\boldsymbol{r},t)\,. \tag{4.32}$$

Thus, we can write, switching to second quantized notation (with V denoting the total volume),

$$\hat{H}'(t) = -\frac{V}{c}\,\hat{\boldsymbol{\jmath}}_{-\boldsymbol{q}}(t)\cdot \boldsymbol{A}_{\boldsymbol{q}}(t) = \frac{iV}{\omega}\,\hat{\boldsymbol{\jmath}}_{-\boldsymbol{q}}(t)\cdot \boldsymbol{E}_{\boldsymbol{q}}\mathrm{e}^{-i\omega t}\,. \tag{4.33}$$

Using (D.21) and assuming that the response is restricted to the same wave vector as the perturbing field, i.e., for not too short wavelengths, the paramagnetic linear response current reads (denoting directions as α or β),

$$\langle \hat{\jmath}_{\boldsymbol{q},\alpha}(\omega)\rangle_A = -\frac{iV}{\omega+i0^+}E_{\boldsymbol{q},\beta}(\omega)\,\langle\langle \hat{\jmath}_{\boldsymbol{q},\alpha},\hat{\jmath}_{-\boldsymbol{q},\beta}\rangle\rangle\,(\omega+i0^+) \tag{4.34}$$

From (4.4), (4.30), and (D.22) we get the general expression for continuum systems[13]

$$\sigma_{\alpha\beta}(\boldsymbol{q},\omega) = \frac{V}{\hbar(\omega+i0^+)}\int_0^\infty \mathrm{d}t\,\mathrm{e}^{i(\omega+i0^+)t}\langle[\hat{\jmath}^\dagger_{\boldsymbol{q},\alpha}(t),\hat{\jmath}_{\boldsymbol{q},\beta}(0)]\rangle + i\frac{n_0e^2}{m(\omega+i0^+)}\delta_{\alpha\beta} \tag{4.35}$$

Here, the time dependence of the operator is in the interaction picture (see Sec. D.2) and the long wavelength limit has been used to replace $n(\boldsymbol{r},t)$ by the average density n_0. In the optical limit $\boldsymbol{q}\to 0$, the current density operator $\hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{\jmath}}^\dagger$ is hermitian. For systems with cubic symmetry, the current flow in perpendicular directions is then uncorrelated and $\sigma_{\alpha\beta}(\boldsymbol{0},\omega)$ reduces to the scalar $\sigma(\omega)$. Equation (4.35) is correct both for $T=0$ (then the outer brackets indicate the ground state expectation value) and for $T>0$ (when the brackets imply thermal averaging) and for longitudinal as well as transverse electric fields.

Since the susceptibility, i.e., the integral in (4.35) falls off as $1/\omega$ or faster [cf. (D.25)], the high-frequency limit of $\sigma(\omega)$ is determined by the second, diamagnetic term which is proportional to the squared free plasma frequency $\omega_\mathrm{p}^2 = 4\pi n_0 e^2/m$. Thus, the frequency sum (4.19) is unchanged by the interactions of electrons, both with the lattice ions and among themselves,

$$\int_0^\infty \mathrm{d}\omega\,\mathrm{Re}\,\sigma_{\alpha\beta}(\omega) = \frac{1}{2}\int_{-\infty}^\infty \mathrm{d}\omega\frac{n_0e^2}{m}\pi\delta(\omega)\,\delta_{\alpha\beta} = \frac{\omega_\mathrm{p}^2}{8}\,\delta_{\alpha\beta} \tag{4.36}$$

[12] Note that the quadratic term in H had to be considered for deriving the correct expression for the full current.

[13] Note that (4.35) is valid in Gaussian units. For SI units, its right hand side has to be divided by $4\pi\epsilon_0$.