In the noninteracting limit, where correlations vanish, $\operatorname{Re}\sigma(\omega)$ reduces to the diamagnetic δ-function contributions. Interactions shift weight from $\omega = 0$ to finite frequencies without changing the total weight. Thus, the full universal f-sum rule is indeed found to be valid, at least within the (zeroth order) Born-Oppenheimer approximation and in the limit that optical absorption decays below a cutoff energy E with $\frac{E}{\hbar c} \ll \frac{\pi}{a}$.

4.2.2 Kubo Formalism on a Lattice

In Wannier representation, the vector potential can be treated (Kohn, 1964; Shastry and Sutherland, 1990; Scalapino, White, and Zhang, 1992; Dagotto, 1994) by a gauge transformation of the Wannier wave functions $\Phi_l(\boldsymbol{r})$ centered at site $\boldsymbol{R}_l$,

$$\Phi_l(\boldsymbol{r}) \mapsto \tilde{\Phi}_l(\boldsymbol{r}, t) = \left[\exp\left(i\frac{e}{c\hbar}\int\limits_{\boldsymbol{R}_0}^{\boldsymbol{R}_l} \mathrm{d}\boldsymbol{r} \cdot \boldsymbol{A}(\boldsymbol{r}', t)\right)\right]\Phi_i(\boldsymbol{r}), \tag{4.37}$$

where the free choice of $\boldsymbol{R}_0$ determines the arbitrary overall phase factor. In the long-wavelength limit $\boldsymbol{q} \to 0$, the vector potential can be taken out of the integral. Then, the change in the Hamiltonian consists of multiplying the hopping elements defined in (1.10) by so-called Peierls phase factors,[14]

$$t_{ij} = t^0_{ij} \exp\left(-i\frac{e}{c\hbar}(\boldsymbol{R}_i - \boldsymbol{R}_j) \cdot \boldsymbol{A}\right). \tag{4.38}$$

Note that only hopping elements with components parallel to the vector field $\boldsymbol{A}$ are modified. In comparing (4.38) with (D.3) we identify the additional factor as $\exp(-i\Delta S/\hbar)$, where ΔS is the field contribution to the action for the world line connecting the lattice sites in absence of a scalar potential ϕ.

Before considering the general case, we specialize on the more commonly treated case of the hypercubic tight-binding model with nearest-neighbor hopping, where the unperturbed kinetic energy is given by $\hat{K}^0 = -t\sum_{\langle ij\rangle,\sigma}\left(\hat{c}^\dagger_{i\sigma}\hat{c}_{j\sigma} + \hat{c}^\dagger_{j\sigma}\hat{c}_{i\sigma}\right)$. Expanding the phase factors to lowest orders in $\boldsymbol{A}$,[15] we obtain the modified kinetic energy:

$$\hat{K} = \hat{K}^0 + \sum_{\alpha=1}^{d}\big(-\frac{V}{c}\hat{\jmath}_\alpha A_\alpha + \frac{e^2a^2}{2\hbar^2c^2}\hat{K}^0_\alpha A^2_\alpha + \dots\big) \tag{4.39}$$

where the lattice paramagnetic current density operator and the kinetic energy in

[14] While, in general, the interaction matrix elements defined in (1.11) also acquire Peierls phases, on-site elements and intersite contributions of the density-density type remain unchanged.

[15] We need quadratic order in order to retain the linear contribution to the current operator, $\hat{J}_\alpha$.