where $\boldsymbol{k}_\perp$ is the projection of $\boldsymbol{k}$ on directions perpendicular to $\boldsymbol{e}_x$. Since the n^{th} term in (4.46) contains hopping to n^{th}-nearest neighbors, we can use straightforward generalizations of (4.39) and (4.42) to find for the paramagnetic current density operator $\hat{\jmath}_x$ and diamagnetic current density operator $\hat{\jmath}_x^{\text{dia}}$, respectively,

$$\hat{\jmath}_x = -\frac{ea_x}{V\hbar}\sum_{\boldsymbol{k},\nu,\sigma}\hat{n}_{\boldsymbol{k},\nu,\sigma}\sum_{n=1}^{\infty} n f_{\nu,n}(\boldsymbol{k}_\perp)\sin(nk_x) = \frac{e}{V\hbar}\sum_{\boldsymbol{k},\nu,\sigma}\hat{n}_{\boldsymbol{k},\nu,\sigma}\frac{\partial}{\partial k_x}\epsilon_{\boldsymbol{k},\nu} \tag{4.47}$$

$$\hat{\jmath}_x^{\text{dia}} = \frac{e^2A}{Vc\hbar^2}\sum_{\boldsymbol{k},\nu,\sigma}\hat{n}_{\boldsymbol{k},\nu,\sigma}\frac{\partial^2}{\partial k_x^2}\epsilon_{\boldsymbol{k},\nu} \tag{4.48}$$

We, thus, obtain for the conductivity,

$$\boxed{\begin{aligned}\sigma_{\alpha\alpha}(\omega) &= \frac{ie^2}{V\hbar^2(\omega+i0^+)}\sum_{\nu,\nu',\sigma,\sigma'}\sum_{\boldsymbol{k},\boldsymbol{k}'} v_{\boldsymbol{k},\nu,\alpha}v_{\boldsymbol{k}',\nu',\alpha}\langle\langle\hat{n}_{\boldsymbol{k},\nu,\sigma};\hat{n}_{\boldsymbol{k}',\nu',\sigma'}\rangle\rangle(\omega+i0^+)\\ &\quad+\frac{ie^2}{V\hbar^2(\omega+i0^+)}\sum_{\boldsymbol{k},\nu,\sigma}\langle\hat{n}_{\boldsymbol{k},\nu,\sigma}\rangle\frac{\partial^2}{\partial k_\alpha^2}\epsilon_{\boldsymbol{k},\nu}\,,\end{aligned}} \tag{4.49}$$

where $v_{\boldsymbol{k},\nu,\alpha} = \frac{\partial}{\partial k_\alpha}\epsilon_{\boldsymbol{k},\nu}$ is the α component of the Fermi velocity of band ν for momentum $\boldsymbol{k}$. Consequently, the f-sum rule reads

$$\boxed{\int_0^\infty \mathrm{d}\omega\,\sigma_{\alpha\alpha}(\omega) = \frac{\pi e^2}{2V\hbar^2}\sum_{\boldsymbol{k},\nu,\sigma}\langle\hat{n}_{\boldsymbol{k},\nu,\sigma}\rangle\frac{\partial^2}{\partial k_\alpha^2}\epsilon_{\boldsymbol{k},\nu}\,.} \tag{4.50}$$

For a full electronic model, we can use the free dispersion for a single band with $\frac{\partial^2\epsilon_{\boldsymbol{k}}}{\partial k^2} = \frac{\hbar^2}{m}$ which leads to the free diamagnetic term and, thus, to the free sum rule (4.21), i.e., $\int\mathrm{d}\omega\,\sigma_{\alpha\alpha}(\omega) = \frac{\omega_{\text{p}}^2}{8}$ with $\omega_{\text{p}}^2 = \frac{4\pi ne^2}{m}$.

4.2.3 General Confirmation of the f-sum Rule

While the fact that the paramagnetic current will not contribute to the f-sum rule follows on quite general grounds from (D.25), we can drop the condition of a bounded spectrum used in Sec. D.2. In the course of re-deriving the sum rule (4.50), we will also identify a contribution to the paramagnetic current which cancels the diamagnetic current. In this subsection, only the single-band case is considered explicitly.

For the Hamiltonian (4.45), the polarization operator $\hat{\boldsymbol{P}} = \sum_i \boldsymbol{r}_i\hat{n}_i$ commutes with the interaction term

$$[\hat{H}_{\text{ee}},\hat{\boldsymbol{P}}] = \sum_i \boldsymbol{r}_i[\hat{H}_{\text{ee}},\hat{n}_i] = 0\,. \tag{4.51}$$

and can be used for expressing the current density operators:

$$\hat{\jmath}_\alpha = \frac{e}{V}\frac{\mathrm{d}}{\mathrm{d}t}\hat{P}_\alpha = -\frac{ie}{V\hbar}[\hat{P}_\alpha,\hat{K}] \tag{4.52}$$

$$\hat{\jmath}_\alpha^{\text{dia}} = \frac{e^2A}{Vc\hbar^2}[\hat{P}_\alpha,[\hat{P}_\alpha,\hat{K}]]\,. \tag{4.53}$$