Thus, we can rewrite the current-current correlation function,

$$\begin{aligned}
\langle\langle \hat{\jmath}_\alpha, \hat{\jmath}_\alpha \rangle\rangle(\omega) &= \frac{ie}{V\hbar}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{i\omega t}\langle[\frac{\mathrm{d}}{\mathrm{d}t}\hat{P}_\alpha(t), \hat{\jmath}_\alpha(0)]\rangle && (4.54)\\
&= \frac{ie}{V\hbar}\left(\mathrm{e}^{i\omega t}\langle[\hat{P}_\alpha(t), \hat{\jmath}_\alpha(0)]\rangle\Big|_0^\infty - \int_0^\infty \mathrm{d}t\,(i\omega)\mathrm{e}^{i\omega t}\langle[\hat{P}_\alpha(t), \hat{\jmath}_\alpha(0)]\rangle\right) && (4.55)\\
&= -\frac{ie}{V\hbar}\langle[\hat{P}_\alpha, \hat{\jmath}_\alpha]\rangle + \frac{e\,\omega}{V\hbar}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{i\omega t}\langle[\hat{P}_\alpha(t), \hat{\jmath}_\alpha]\rangle && (4.56)
\end{aligned}$$

The operator in the first term, $-\frac{ie}{V\hbar}[\hat{P}_\alpha, \hat{\jmath}_\alpha] = -\frac{e^2}{V^2\hbar^2}[\hat{P}_\alpha, [\hat{P}_\alpha, \hat{K}]]$, cancels the diamagnetic contribution to $\sigma_{\alpha\alpha}(\omega)$. Thus, the f-sum reads,[18]

$$\begin{aligned}
\int_0^\infty \mathrm{d}\omega\, \mathrm{Re}\, \sigma_{\alpha\alpha}(\omega) &= \frac{1}{2}\frac{ie}{\hbar}\int_0^\infty \mathrm{d}t\, 2\pi\delta(t)\langle[\hat{P}_\alpha(t), \hat{\jmath}_\alpha]\rangle && (4.57)\\
&= \frac{\pi e^2}{2V\hbar^2}\langle[\hat{P}_\alpha, [\hat{P}_\alpha, \hat{K}]]\rangle\,, && (4.58)
\end{aligned}$$

which reduces to (4.50) since $[\hat{P}_\alpha, [\hat{P}_\alpha, \hat{K}]] = \sum_{\boldsymbol{k},\sigma} \hat{n}_{\boldsymbol{k},\sigma}\frac{d^2}{dk_\alpha^2}\epsilon_{\boldsymbol{k}}$.

From (4.56), we can also express the total optical conductivity as

$$\sigma_{\alpha\alpha}(\omega) = e\langle\langle \hat{P}_\alpha, \hat{\jmath}_\alpha \rangle\rangle(\omega)\,. \quad (4.59)$$

This form makes the qualitative behavior seen in Fig. 4.5 physically plausible: In the presence of (small) interactions, quasiparticles acquire a finite lifetime and, thus, correlations generically decay in real time which leads to $\sigma(\omega)$ being finite (i.e., not infinite) at all ω. At $T = 0$, however, quasiparticles directly at the Fermi surface are still stable and may give rise to a (reduced) Drude δ-peak. Within Fermi liquid theory, the reduction factor is Z, the quasiparticle weight. At finite T, all lifetimes are finite (but still large near the Fermi surface) so that the Drude peak broadens with increasing interaction until both Fermi liquid theory breaks down and the Drude peak may disappear in the incoherent background or vanish when a gap opens. Assuming, on the other hand, that $\sigma(\omega)$ remains finite at all ω, one can as well work with the real part of the paramagnetic contribution for $\omega > 0$ as given by (4.49) and obtain σ_{dc} as $\lim_{\omega\to 0}\sigma(\omega)$. In fact, in numerical calculations at $T = 0$ the Drude weight is often obtained via the f-sum rule (Dagotto, 1994).

[18] Strictly speaking, the integral on the right hand side of (4.57) is undefined. For a more rigorous proof, one needs to keep a "small" real $\eta > 0$, use $\mathrm{Re}\int_0^\infty \mathrm{d}\omega \mathrm{e}^{i\omega(t+i\eta)} = \eta/(\eta^2 + t^2)$ and the fact that $\mathrm{Im}\,\langle i[\hat{A}, \hat{B}]\rangle = 0$ for Hermitian operators $\hat{A}$, $\hat{B}$. Then, (4.58) can be derived from $\lim_{\eta\to 0}\mathrm{Re}\int_0^\infty \mathrm{d}\omega\int_0^\infty \mathrm{d}t \mathrm{e}^{i\omega(t+i\eta)}f(t) = f(0^+)\lim_{\eta\to 0}\int_0^\infty \mathrm{d}t\eta/(\eta^2+t^2) = f(0^+)\pi/2$ for a real function $f(t)$.