Thus, vertex corrections do not contribute in $d \to \infty$ and (4.60) reduces to the bubble contribution (with inverse temperature β and Matsubara frequencies $i\omega_n$)

$$\sigma_{xx}(i\nu) = \frac{e^2}{V\hbar^2\nu} T \sum_{\boldsymbol{k},\sigma,\omega_n} v_{\boldsymbol{k}_x}^2 G_{\boldsymbol{k}\sigma}(i\omega_n)\, G_{\boldsymbol{k}\sigma}(i\omega_n + i\nu)\,. \tag{4.61}$$

The momentum independence[21] of $\Sigma(\omega)$ can now be used for replacing the momentum sum by an energy integral (where N denotes number of sites)

$$\sigma_{xx}(i\nu) = 2\frac{e^2}{\hbar^2\nu}\frac{N}{V}\int_{-\infty}^{\infty} \mathrm{d}\epsilon\, \tilde{\rho}_{xx}(\epsilon)\, T \sum_n G_\epsilon(i\omega_n) G_\epsilon(i\omega_n + i\nu)\,. \tag{4.62}$$

Here, we have introduced [cf. (3.11)]

$$G_\epsilon(i\omega_n) \equiv G_{\epsilon_{\boldsymbol{k}},\sigma}(i\omega_n) = \frac{1}{i\omega_n - \epsilon_{\boldsymbol{k}} - \Sigma_\sigma(i\omega_n)} \tag{4.63}$$

and

$$\tilde{\rho}_{xx}(\epsilon) := \frac{1}{N}\sum_{\boldsymbol{k}} v_{\boldsymbol{k}_x}^2 \delta(\epsilon - \epsilon_{\boldsymbol{k}})\,. \tag{4.64}$$

and specialized on the paramagnetic case. The factor of 2 in (4.62) is for spin degeneracy. Note that $\tilde{\rho}_{xx}(\epsilon) = \mathcal{O}(1/d)$; in the isotropic case, we have $d\tilde{\rho}_{xx}(\epsilon) = \tilde{\rho}(\epsilon) := \frac{1}{N}\sum_{\boldsymbol{k}} |\boldsymbol{v}_{\boldsymbol{k}}|^2 \delta(\epsilon - \epsilon_{\boldsymbol{k}})$. Analytic continuation of (4.62) yields

$$\boxed{\mathrm{Re}\,\sigma_{xx}(\omega) = \sigma_0 \int_{-\infty}^{\infty} \mathrm{d}\epsilon\, \tilde{\rho}_{xx}(\epsilon) \int_{-\infty}^{\infty} \mathrm{d}\omega'\, A_\epsilon(\omega') A_\epsilon(\omega' + \omega) \frac{n_{\mathrm{f}}(\omega') - n_{\mathrm{f}}(\omega + \omega')}{\omega}\,,} \tag{4.65}$$

where

$$A_\epsilon(\omega) = -\frac{1}{\pi}\mathrm{Im}\, G_\epsilon(\omega)\,, \qquad n_{\mathrm{f}}(\omega) = \left(1 + \mathrm{e}^{\beta(\omega-\mu)}\right)^{-1}, \tag{4.66}$$

and

$$\sigma_0 := \frac{2\pi e^2}{\hbar^2}\frac{N}{V}\,. \tag{4.67}$$

We note that the limit of infinite dimensionality enters (4.65) only via the locality of the self-energy and of the irreducible vertex function. Using both properties as an approximation, (4.65) may therefore also be applied in finite dimensions. For numerical results, we will set e, $\hbar$, and the lattice spacing a to unity which corresponds to $\sigma_0 = 2\pi$.

[21] If $\Sigma(\omega)$ was momentum-dependent, the momentum sum in the general bubble-sum expression (4.61) could not be replaced by an energy integral. For piecewise constant $\Sigma(\omega)$ like in the DCA, however, a generalization of (4.152) could be applied on each patch. The individual contributions for each value of the coarse-grained momentum would then have to be summed up for the total optical conductivity.