4.3.1 Optical Conductivity for the Hypercubic Lattice

For a hypercubic lattice with NN hopping, $\tilde{\rho}_{xx}(\epsilon)$ is proportional to the DOS,

$$\tilde{\rho}^{\rm hc}_{xx}(\epsilon) = \frac{t^{*2}a^2}{d}\rho^{\rm hc}(\epsilon) \tag{4.68}$$

We will here first reproduce Pruschke et al.'s (1993) proof using the same Fourier transform technique that had been used before for calculating the (Gaussian) DOS of the hc lattice (Müller-Hartmann, 1989*a*) and then derive a shorter version which is more general. Using $\epsilon_{\boldsymbol{k}} = -2t\sum_{\alpha=1}^{d}\cos(k_\alpha)$ and $\frac{\partial\epsilon_{\boldsymbol{k}}}{\partial k_\alpha} = 2t\sin(k_\alpha)$, which corresponds to lattice spacing $a=1$, $\tilde{\rho}^{\rm hc}_{xx}(\epsilon)$ transforms as follows,

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{d}\epsilon\,\tilde{\rho}_{xx}(\epsilon)\,\mathrm{e}^{is\epsilon} &= \left[\int_{-\pi}^{\pi}\frac{\mathrm{d}k}{2\pi}\mathrm{e}^{-2ist\cos(k)}\right]^{d-1}\left[\int_{-\pi}^{\pi}\frac{\mathrm{d}k}{2\pi}4t^2\sin^2(k)\mathrm{e}^{-2ist\cos(k)}\right] && (4.69)\\
&= 2t^2\left[J_0(2st)\right]^d\left(1+\frac{J_2(2st)}{J_0(2st)}\right) && (4.70)\\
&= 2t^2\exp\left[d\ln\left(1-\frac{(2st)^2}{4}+\cdots\right)+\ln\left(1+\frac{(2st)^2}{8}+\cdots\right)\right]\\
&= 2t^2\exp\left[-\frac{(st^*)^2}{2}+\mathcal{O}(\frac{1}{d})\right]\,. && (4.71)
\end{aligned}$$

Keeping only leading terms, transforming back, and determining the dependence on a from scaling, one obtains (using $Z=2d$)

$$\tilde{\rho}_{xx}(\epsilon) = 2t^2a^2\rho(\epsilon)\,, \tag{4.72}$$

which confirms (4.68). Thus, the conductivity reads,

$$\boxed{\begin{aligned}
\mathrm{Re}\,\sigma^{\rm hc}_{xx}(\omega) &= \frac{2\pi e^2a^2t^{*2}}{d\hbar^2}\frac{N}{V}\int_{-\infty}^{\infty}\mathrm{d}\epsilon\,\rho(\epsilon)\\
&\times\int_{-\infty}^{\infty}\mathrm{d}\omega'\,A_\epsilon(\omega')A_\epsilon(\omega'+\omega)\,\frac{n_{\rm f}(\omega')-n_{\rm f}(\omega+\omega')}{\omega}\,.
\end{aligned}} \tag{4.73}$$

While the above approach using Bessel functions $J_\nu(x)$ has the advantage that it allows for the calculation of corrections terms of $\mathcal{O}(1/d)$ and beyond, the $d\to\infty$ result can be obtained more directly: Let us assume that the total noninteracting kinetic energy separates into a contribution depending only on hopping in the current direction and a contribution which does not involve hopping in that direction and that the latter contribution dominates the bandwidth,

$$\epsilon_{\boldsymbol{k}} = \epsilon^{\perp}_{\boldsymbol{k}_\perp} + \epsilon^{x}_{k_x}\,,\quad \frac{\sqrt{\langle(\epsilon^x_{k_x}-\langle\epsilon^x_{k_x}\rangle)^2\rangle}}{\sqrt{\langle(\epsilon_{\boldsymbol{k}}-\langle\epsilon_{\boldsymbol{k}}\rangle)^2\rangle}} = \mathcal{O}(1/d)\,. \tag{4.74}$$