potential μ, and spectral function $A_{\epsilon,\sigma}(\omega)$ by

$$n_{\epsilon,\sigma} := \langle \hat{n}_{\epsilon,\sigma} \rangle = \int_{-\infty}^{\infty} \mathrm{d}\omega \, n_{\mathrm{f}}(\omega - \mu) A_{\epsilon,\sigma}(\omega) \,. \tag{4.79}$$

In the paramagnetic case, $n_\epsilon \equiv n_{\epsilon,\sigma}$. While the expression (4.77) was derived for the noninteracting limit, and, thus, explicitly only determines the weight of the Drude peak, one might suspect that it remains valid in connection with (4.79) in the interacting case. This would amount to a generalization of the hc sum rule (4.44), where (for $t^* = 1$ and $a = 1$), $\tilde{\rho}(\epsilon) = \rho(\epsilon) = \mathrm{e}^{-\epsilon^2/2}$ and $\langle \tilde{\rho}'(\epsilon)/\rho(\epsilon) \rangle = \langle -\epsilon \rangle$ is minus the kinetic energy. For a proof of (4.77) in the general case, we will show that the frequency dependent part in (4.65), denoted as $I(\epsilon)$, sums up to $-\mathrm{d}n_\epsilon/\mathrm{d}\epsilon$,

$$\begin{aligned}
I(\epsilon) &= \int_{-\infty}^{\infty} \mathrm{d}\omega \int_{-\infty}^{\infty} \mathrm{d}\omega' A_\epsilon(\omega') A_\epsilon(\omega' + \omega) \frac{n_{\mathrm{f}}(\omega') - n_{\mathrm{f}}(\omega' + \omega)}{\omega} && (4.80)\\
&= \int_{-\infty}^{\infty} \mathrm{d}\omega'' \int_{-\infty}^{\infty} \mathrm{d}\omega' \, A_\epsilon(\omega') A_\epsilon(\omega'') \frac{n_{\mathrm{f}}(\omega') - n_{\mathrm{f}}(\omega'')}{\omega'' - \omega'} && (4.81)\\
&= \int_{-\infty}^{\infty} \mathrm{d}\omega' \, n_{\mathrm{f}}(\omega') A_\epsilon(\omega') \int_{-\infty}^{\infty} \mathrm{d}\omega'' \, \frac{A_\epsilon(\omega'')}{\omega'' - \omega'} \\
&\quad + \int_{-\infty}^{\infty} \mathrm{d}\omega'' \, n_{\mathrm{f}}(\omega'') A_\epsilon(\omega'') \int_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{A_\epsilon(\omega')}{\omega' - \omega''} && (4.82)\\
&= \frac{2}{\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega \, n_{\mathrm{f}}(\omega) \, \mathrm{Re}\, G_\epsilon(\omega) \, \mathrm{Im}\, G_\epsilon(\omega) && (4.83)\\
&= -\frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega \, n_{\mathrm{f}}(\omega) \, \frac{\mathrm{d}}{\mathrm{d}\epsilon} \, \frac{\mathrm{Im}\, \Sigma(\omega)}{(\omega - \epsilon - \mathrm{Re}\, \Sigma(\omega))^2 + \mathrm{Im}\, \Sigma(\omega)^2} && (4.84)\\
&= -\frac{\mathrm{d}}{\mathrm{d}\epsilon} \, n_\epsilon \,. && (4.85)
\end{aligned}$$

Thus, the f-sum rule for the optical conductivity within the DMFT, i.e., within the bubble approximation with a $\boldsymbol{k}$-independent self-energy reads

$$\begin{aligned}
\int_{0}^{\infty} \mathrm{d}\omega \, \sigma_{xx}(\omega) &= \frac{\sigma_0}{2} \int_{-\infty}^{\infty} \mathrm{d}\epsilon \, \tilde{\rho}_{xx}(\epsilon) \Big(-\frac{dn_\epsilon}{d\epsilon} \Big) && (4.86)\\
&= \frac{\sigma_0}{2} \int_{-\infty}^{\infty} \mathrm{d}\epsilon \, \tilde{\rho}'_{xx}(\epsilon) \, n_\epsilon = \frac{\sigma_0}{4} \, \langle \frac{\tilde{\rho}'_{xx}(\epsilon)}{\rho(\epsilon)} \rangle \,. && (4.87)
\end{aligned}$$

This sum rule is of great practical value as a check of numerical results. Note that it cannot only be evaluated from real-frequency data, but also directly on the imaginary