where we have exploited the fact that the energy of the hypercubic lattice separates into contributions depending on k_x and on components of $\boldsymbol{k}$ perpendicular to the x direction, respectively. Since the former is of order $1/d$ it could be neglected in the fourth line of (4.92). In an analogous fashion one can compute the sum in (4.50)

$$\lim_{d\to\infty}\sum_{\boldsymbol{k},\sigma}\langle\hat{n}_{\boldsymbol{k}}\rangle\, d\frac{\partial^2\epsilon_{\boldsymbol{k}}}{\partial k_x^2} = 2\int_{-\infty}^{\infty} \mathrm{d}\epsilon\,\rho(\epsilon)\left[\mathcal{F}''(\mathcal{F}^{-1}(\epsilon)) - \mathcal{F}'(\mathcal{F}^{-1}(\epsilon))\mathcal{F}^{-1}(\epsilon)\right]\langle\hat{n}_\epsilon\rangle \tag{4.93}$$

Finally, since $\tilde{\rho}'(\epsilon)$ is related with the second derivative of the dispersion,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\epsilon}\tilde{\rho}(\epsilon) &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big(\big(\mathcal{F}'(\mathcal{F}^{-1}(\epsilon))\big)^2\rho(\epsilon)\Big)\\
&= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\left(\mathcal{F}'\big(\mathcal{F}^{-1}(\epsilon)\big)\,\rho^{\mathrm{hc}}\big(\mathcal{F}^{-1}(\epsilon)\big)\right)\\
&= \mathcal{F}'\big(\mathcal{F}^{-1}(\epsilon)\big)\,\big(\rho^{\mathrm{hc}}\big)'\big(\mathcal{F}^{-1}(\epsilon)\big)\,\big(\mathcal{F}^{-1}(\epsilon)\big)'\\
&\quad +\mathcal{F}''\big(\mathcal{F}^{-1}(\epsilon)\big)\,\big(\mathcal{F}^{-1}(\epsilon)\big)'\,\rho^{\mathrm{hc}}\big(\mathcal{F}^{-1}(\epsilon)\big)\\
&= \big(\rho^{\mathrm{hc}}\big)'\big(\mathcal{F}^{-1}(\epsilon)\big) + \frac{\mathcal{F}''\big(\mathcal{F}^{-1}(\epsilon)\big)}{\mathcal{F}'\big(\mathcal{F}^{-1}(\epsilon)\big)}\rho^{\mathrm{hc}}\big(\mathcal{F}^{-1}(\epsilon)\big)\\
&= \left[\mathcal{F}''\big(\mathcal{F}^{-1}(\epsilon)\big) - \mathcal{F}^{-1}(\epsilon)\mathcal{F}'\big(\mathcal{F}^{-1}(\epsilon)\big)\right]\rho(\epsilon)
\end{aligned} \tag{4.94}$$

we can reformulate the general DMFT sum rule (4.87) in terms of the dispersion:

$$\boxed{\begin{aligned}
\int_0^\infty \mathrm{d}\omega\,\sigma_{xx}(\omega) &\overset{(4.87)}{=} \frac{\sigma_0}{4d}\Big\langle\frac{\tilde{\rho}'(\epsilon)}{\rho(\epsilon)}\Big\rangle\\
&\overset{(4.94)}{=} \frac{\sigma_0}{4d}\left\langle\mathcal{F}''\big(\mathcal{F}^{-1}(\epsilon)\big) - \mathcal{F}^{-1}(\epsilon)\mathcal{F}'\big(\mathcal{F}^{-1}(\epsilon)\big)\right\rangle\\
&\overset{(4.93)}{=} \frac{\sigma_0}{4}\sum_{\boldsymbol{k},\sigma}\langle\hat{n}_{\boldsymbol{k}\sigma}\rangle\frac{\partial^2\epsilon_{\boldsymbol{k}}}{\partial k_x^2}\,.
\end{aligned}} \tag{4.95}$$

Since (4.95) is equivalent to (4.50) for a single band, we have demonstrated the consistency of our formalism.

4.4 Optical Conductivity for the Bethe Lattice

As discussed in Sec. 2.2, the Bethe "lattice" is more accurately labeled as a tree (a special planar graph) since it is does not constitute a regular lattice. In fact, the periodicity of a regular lattice and the associated graph formed by, e.g., its nearest-neighbor bonds generally imply the existence of an infinite number of inequivalent loops for $Z > 2$ while the Bethe "lattice" is defined by the complete absence of loops (and a constant number of nearest neighbors relative to each site). Due to its definition in topological terms only, concepts like directions, angles, and distances