For the kinetic energy, we can write

$$\hat{T} = -2t \sum_{y} \sum_{\alpha_y} \hat{c}^{\dagger}_{y,\alpha_y} \big(\hat{c}_{y-1,f(\alpha_y)} + \sum_{i=1}^{Z-1} \hat{c}_{y+1,g(\alpha_y,i)} \big) \,, \tag{4.99}$$

where $f(\alpha_y)$ maps onto the intra-level coordinate of the "root" (on level $y-1$) of site (y, α_y) . Conversely, $g(\alpha_y, i)$ maps onto the intra-level coordinate of one of the $(Z-1)$ "children" on the higher level of the same site.[27] We obtain for the eigenvalue equation

$$\begin{aligned} \hat{T}|\epsilon\rangle &= -2t \sum_{x} \big(\gamma_\epsilon^{x-1} + (Z-1)\gamma_\epsilon^{x+1}\big) \sum_{\alpha_x} |x, \alpha_x\rangle & (4.100) \\ &= -2t\big(\gamma_\epsilon^{-1} + (Z-1)\gamma_\epsilon\big)|\epsilon\rangle \,. & (4.101) \end{aligned}$$

By definition, the coefficient if front of $|\epsilon\rangle$ must equal ϵ; inverting this equation, we obtain

$$\gamma_\epsilon = \frac{-\epsilon \pm \sqrt{\epsilon^2 - 4(Z-1)t^2}}{2(Z-1)t} \,. \tag{4.102}$$

Expressing the current operator[28] in a form analogous to the kinetic energy,

$$\hat{\jmath} = iet \sum_{y} \sum_{\alpha_y} \hat{c}^{\dagger}_{y,\alpha_y} \big(- \hat{c}_{y-1,f(\alpha_y)} + \sum_{i=1}^{Z-1} \hat{c}_{y+1,g(\alpha_y,i)} \big) \,, \tag{4.103}$$

the associated eigenvalue equation reads,

$$\begin{aligned} \hat{\jmath}|\epsilon\rangle &= iet\big(- \gamma_\epsilon^{-1} + (Z-1)\gamma_\epsilon\big)|\epsilon\rangle & (4.104) \\ &= \pm e\sqrt{4(Z-1)t^2 - \epsilon^2}|\epsilon\rangle \,. & (4.105) \end{aligned}$$

The requirement that the eigenvalues of $\hat{\jmath}$ have to be real fixes the band edges, $|\epsilon| \le 2t\sqrt{Z-1}$.[29] The square root in (4.105) is to be interpreted as $\langle v_{\boldsymbol{k}_x}\rangle(\epsilon)$ which implies that in the limit of $Z \to \infty$,

$$\tilde{\rho}_{xx}(\epsilon) = (4t^{*2} - \epsilon^2)\rho(\epsilon) \,. \tag{4.106}$$

Applying the general expression (2.24) specifically for the semi-elliptic DOS (for $t^* = 1$) then leads to

$$\begin{aligned} \tilde{\rho}_{xx}(\epsilon) &= \frac{1}{2\pi}\big(4 - \epsilon^2\big)^{3/2} & (4.107) \\ \int_0^\infty \mathrm{d}\omega\, \sigma_{xx}(\omega) &= \frac{\sigma_0}{4}\langle \frac{\tilde{\rho}'_{xx}(\epsilon)}{\rho(\epsilon)}\rangle = \frac{\sigma_0}{4}\langle -3\epsilon\rangle \,, & (4.108) \end{aligned}$$

[27] More formally, we can demand that the mapping $(y, \alpha_y, i) \to (y+1, \alpha_{y+1}) \equiv (y+1, g(\alpha_y, i))$ is injective and that $f(g(\alpha_y, i)) = \alpha_y$ for all y, α_y, and $i \in \{1, \ldots, Z-1\}$.

[28] Since no volume can be defined, we consider here the full current instead of the current density. Furthermore, we set $a = 1$ and $\hbar = 1$.

[29] Freericks links the restriction of ϵ to be within the band edges to the requirement that the states are extended. See also the discussion below.