where for the sum rule we have used (4.87). Since only one direction is well defined, we can identify $\sigma_{xx}(\omega)$ with $\mathrm{Tr}\left\{\sigma_{xx}(\omega)\right\}$ and find that the corresponding sum rule exceeds the "usual" hypercubic case by a factor of 3. While, in view of the preceding section, we are not surprised to find different sum rules (in terms of specific expectation values) for different lattices, one might view the appearance of the factor of 3 as a clue to some inconsistency in this particular case.

We will show in the following that the preceding derivation (4.98) - (4.108) is incorrect for the Bethe lattice and instead applies to some effective one-dimensional lattice. In order to establish operator identities or, in this case, that the eigenvalue of the current operator can always be expressed in terms of the eigenvalue of the kinetic energy [cf. (4.104)], one needs to demonstrate such a relationship for a complete basis. However, the class of wave functions represented by the ansatz (4.98) is not complete for $Z > 2$. This is easily seen from the fact that, according to this ansatz, the occupation is constant within each level, $\langle \hat{n}_{x,\alpha_x} \rangle = \langle \hat{n}_{x,\alpha'_x} \rangle$ while each single hopping process induces an intraplanar variation of the occupancy.[30] Therefore, Freericks' derivation at best applies to some subspace of the full Hilbert space only.[31] Let us characterize this reduced Hilbert space by finding an orthogonal basis for a section of the Bethe lattice with an arbitrary but finite number L of levels. Setting

$$\gamma_\epsilon = \frac{\tilde{\gamma}_\epsilon}{\sqrt{Z-1}}, \tag{4.109}$$

we can write

$$\begin{aligned} \langle \epsilon | \epsilon' \rangle &= N_0 \sum_{x=1}^{L} (Z-1)^x \gamma_\epsilon^x \gamma_{\epsilon'}^{*\,x} = N_0 \sum_{x=1}^{L} (\tilde{\gamma}_\epsilon \tilde{\gamma}_{\epsilon'}^*)^x & (4.110) \\ &= N_0 \sum_{x=1}^{L} \mathrm{e}^{i(\varphi_\epsilon - \varphi_{\epsilon'})x} . & (4.111) \end{aligned}$$

Here, N_0 is the number of sites in level 0. In (4.111), we have used the fact that $\tilde{\gamma}_\epsilon$ is of unit absolute value for physical values of ϵ and have introduced its (real) phase $\varphi_\epsilon = \arg(\tilde{\gamma}_\epsilon)$. For orthogonality, we have to require that the phases differ by integer multiples of $2\pi/L$ which fixes the allowed values of $\tilde{\gamma}_l \equiv \tilde{\gamma}_{\epsilon_l}$ up to some arbitrary phase chosen as zero,

$$\begin{aligned} \tilde{\gamma}_l &= \mathrm{e}^{2\pi i l/L}; \quad l \in \{0, ..., L-1\} & (4.112) \\ \epsilon_l &= -2t\sqrt{Z-1}\,\cos\left(\frac{2\pi l}{L}\right) . & (4.113) \end{aligned}$$

[30] While for any energy eigenstate the total occupation of each level is also constant (which is clearly unphysical since it implies infinite occupation per lattice site for level number $x \to -\infty$), the total occupancy per level can be adjusted independently for each level by choosing a suitable linear combination of eigenfunctions (4.98).

[31] Note that the situation is completely different for wave functions than it is for Green functions. In the latter case, the position where a particle is destroyed singles out a central site. The distance from this site then uniquely defines levels and bond directions (for the unperturbed Hamiltonian). In contrast, the Hilbert space defined in terms of wave functions has to restore the equivalence of all lattice sites.