With these settings, we have $\langle \epsilon_l | \epsilon_{l'} \rangle = N_0 L \delta_{ll'}$. Up to the prefactor $\sqrt{Z-1}$, (4.113) is identical to an enumeration of states of the usual $d = 1$ lattice. Therefore, in the limit $L \to \infty$, the density of states corresponding to the selected set of wave function reads,[32]

$$\rho(\epsilon) = \frac{1}{\pi \sqrt{4t^2(Z-1) - \epsilon^2}} \overset{Z\to\infty}{\longrightarrow} \frac{1}{\pi \sqrt{4t^{*2} - \epsilon^2}} \tag{4.114}$$

Setting $t^* = 1$, we obtain for the sum rule,

$$\tilde{\rho}_{xx}(\epsilon) = (4-\epsilon^2)\rho(\epsilon);\ \rho(\epsilon) = \frac{1}{\pi\sqrt{4-\epsilon^2}} \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\epsilon}\tilde{\rho}_{xx}(\epsilon) = -\frac{\epsilon}{\pi}\left(4-\epsilon^2\right)^{-1/2} \tag{4.115}$$

$$\Rightarrow \int_0^\infty \mathrm{d}\omega\, \sigma_{xx}(\omega) = \frac{\sigma_0}{4}\langle -\epsilon \rangle\,, \tag{4.116}$$

as for the hypercubic lattice. So the factor of 3 was indeed incorrect as the derivation really applied to some effective one-dimensional model.[33] In our view, the inadvertent reduction of the Hilbert space cannot be healed within Freerick's approach since this would involve considering arbitrary level ordering of the lattice sites for a given energy. Even if such an enumeration was possible it seems extremely unlikely that linear combinations can be found which are also eigenstates of the current operator.

Finally, we note that Millis and collaborators also derived an expression for $\sigma(\omega)$ for the Bethe lattice by assuming that the "usual" hc sum rule (4.44) holds (Chattopadhyay, Millis, and Das Sarma, 2000). Consequently, their result agrees with (4.107) and (4.108) up to a factor of 1/3. Since this assumption is also unjustified, there is at present no valid indication at all that an expression of the form $\langle v_{\boldsymbol{k}_x}^2 \rangle(\epsilon) \propto (4-\epsilon^2)$ can be derived for any layout of the Bethe lattice or that the treelike layout of the Bethe lattice (cf. Fig. 4.6 a) is useful for deriving an expression for $\langle v_{\boldsymbol{k}_x}^2 \rangle(\epsilon)$. Therefore, we have to turn to other approaches which will be discussed in the following subsections.

### 4.4.2 Single-Chain Stacked Bethe Lattice

In this section, we will use an alternative real-space picture where the Bethe lattice is laid out in stacked equivalent plains which are connected by a single chain of bonds as shown in Fig. 4.6 c. Once the layout is chosen, the derivation by Logan and his student Stumpf (1999), which we will discuss in the next paragraph, corresponds to a straightforward application of a general real-space expression for the (paramagnetic contribution to the) optical conductivity in high dimensions (Metzner, Schmit, and Vollhardt, 1992). The main virtue of this method is that it leaves the topology of the Bethe lattice intact and is therefore as specific to the Bethe lattice as possible. Its

[32] Note that this expression cannot be correct for any lattice model with hopping parameter $t$ for $Z \neq 2$ since as shown in Sec. 2.2 the variance for NN hopping must always equal $Zt^2$ while from (4.114) it computes to $2(Z-1)t^2$.

[33] Interestingly, the total expression for $\tilde{\rho}_{xx}(\epsilon)$ equals two times the Bethe DOS $\rho(\epsilon)$. So an application of the results of this subsection as an approximation would imply using $\langle v_{\boldsymbol{k}_x}^2 \rangle(\epsilon) = 4$ which up to constant prefactors is equivalent to using the hc formula for the Bethe lattice.