($\boldsymbol{v} = \dot{\boldsymbol{r}}$),

$$\mathcal{L} = -mc^2\sqrt{1-\frac{v^2}{c^2}} + \frac{q}{c}\boldsymbol{A}\cdot\boldsymbol{v} - q\phi. \tag{D.4}$$

Expressing the Hamiltonian,

$$\mathcal{H} = \boldsymbol{v}\cdot\frac{\partial\mathcal{L}}{\partial\boldsymbol{v}} - \mathcal{L} = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} + q\phi, \tag{D.5}$$

in terms of the canonical momentum $\boldsymbol{p}$ (component-wise derivative),

$$\boldsymbol{p} = \frac{\partial\mathcal{L}}{\partial\boldsymbol{v}} = \frac{m\boldsymbol{v}}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{q}{c}\boldsymbol{A} \tag{D.6}$$

one arrives at the final relativistic expression,

$$\begin{aligned}\mathcal{H}(\boldsymbol{p},\boldsymbol{r},t) &= \sqrt{m^2c^4 + c^2\left(\boldsymbol{p}-\frac{q}{c}\boldsymbol{A}(\boldsymbol{r},t)\right)^2} + q\phi(\boldsymbol{r},t) &\text{(D.7)}\\ &\approx mc^2\left(1+\frac{1}{2}\Big(\frac{\boldsymbol{p}-\frac{e}{c}\boldsymbol{A}(\boldsymbol{r},t)}{mc}\Big)^2 - \frac{1}{4}\Big(\frac{\boldsymbol{p}-\frac{e}{c}\boldsymbol{A}(\boldsymbol{r},t)}{mc}\Big)^4\right) + q\phi(\boldsymbol{r},t)\,. &\text{(D.8)}\end{aligned}$$

In the nonrelativistic limit, only the leading nonconstant term needs to be kept,

$$\mathcal{H}(\boldsymbol{p},\boldsymbol{r},t) \approx \mathcal{H} + \frac{1}{2m}\left(\boldsymbol{p}-\frac{q}{c}\boldsymbol{A}(\boldsymbol{r},t)\right)^2 + q\phi(\boldsymbol{r},t). \tag{D.9}$$

The Lorentz force given in (D.1) is easily reproduced from the corresponding Lagrangian.

The potentials $\boldsymbol{A}$, ϕ are not uniquely defined by the physical fields $\boldsymbol{E}$, $\boldsymbol{B}$; the latter remain unchanged under a gauge transformation

$$\begin{aligned}\boldsymbol{A}(\boldsymbol{r},t) &\longrightarrow \boldsymbol{A}'(\boldsymbol{r},t) = \boldsymbol{A}(\boldsymbol{r},t) + \nabla\chi(\boldsymbol{r},t) &\text{(D.10)}\\ \phi(\boldsymbol{r},t) &\longrightarrow \phi'(\boldsymbol{r},t) = \phi(\boldsymbol{r},t) - \frac{1}{c}\frac{\partial\chi(\boldsymbol{r},t)}{\partial t}. &\text{(D.11)}\end{aligned}$$

The Coulomb gauge $\nabla\cdot\boldsymbol{A} = 0$ (also called transverse gauge) is particularly suitable for the study of interacting electron systems, because ϕ acts instantaneously in this case. An oscillatory electric field is then represented by the field $\boldsymbol{A}$ only. Note that this treatment implies an infinitely growing vector potential in the limit $\omega \to 0$ which at least a priori gives reason for caution in connection with linear response theory. The final expressions for the dc conductivity can, however, also be computed directly with the same result (Luttinger, 1964; Mahan, 1990).

D.2 Linear Response Theory

Let a system be subjected to a time-dependent perturbation $F(t)$ which couples through an operator $\hat{A}$,

$$\hat{H}(t) = \hat{H}_0 + \hat{H}_1(t) = \hat{H}_0 - \hat{A}F(t)\,. \tag{D.12}$$