We assume that the function is switched on adiabatically,

$$F(t) \overset{t\to-\infty}{\longrightarrow} 0; \quad \Big|\frac{\mathrm{d}}{\mathrm{d}t}F(t)\Big| < \infty\,. \tag{D.13}$$

Then, the evolution of the density operator for an ensemble of perturbed systems at finite temperature from the unperturbed density operator,

$$\hat{\rho}_0 = \frac{1}{Z_0}\mathrm{e}^{-\beta\hat{H}_0}, \quad Z_0 = \mathrm{Tr}\left\{\mathrm{e}^{-\beta\hat{H}_0}\right\} = \sum_n \mathrm{e}^{-\beta E_n}\,, \tag{D.14}$$

is determined by the von Neumann equation,

$$\hat{\rho}(t=-\infty) = \hat{\rho}_0; \quad i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}(t) = [\hat{H}(t), \hat{\rho}(t)]\,. \tag{D.15}$$

Due to the invariance of the trace with respect to cyclical permutations, expectation values of operators may be equivalently expressed in the Schrödinger or in the interaction picture,

$$\langle\hat{B}\rangle_F := \mathrm{Tr}\left\{\hat{B}\,\hat{\rho}(t)\right\} = \mathrm{Tr}\left\{\hat{B}_I(t)\,\hat{\rho}_I(t)\right\}. \tag{D.16}$$

Here, we denote expectation values in the perturbed ensemble as $\langle\ldots\rangle_F$ and operators in the interaction picture as $\hat{X}_I(t) := \mathrm{e}^{i\hat{H}_0 t/\hbar}\hat{X}\mathrm{e}^{-i\hat{H}_0 t/\hbar}$. The corresponding equation of motion for the density operator plus the initial condition are equivalent to the integral equation

$$\hat{\rho}_I(t) = \hat{\rho}_0 + \frac{i}{\hbar}\int_{-\infty}^{t}\mathrm{d}t'\,[\hat{A}_I(t'), \hat{\rho}_I(t')]F(t')\,, \tag{D.17}$$

which may be solved iteratively. Inserting the unperturbed density operator in the right hand side of (D.17) yields the lowest order (i.e., linear) correction. Using $\mathrm{Tr}\left\{\hat{B}_I(t)[\hat{A}_I(t), \hat{\rho}_0]\right\} = \mathrm{Tr}\left\{\hat{\rho}_0[\hat{B}_I(t), \hat{A}_I(t)]\right\}$ one, thus, obtains to linear order

$$\langle\hat{B}\rangle_F(t) = \langle\hat{B}\rangle_0 + \int_{-\infty}^{\infty}\mathrm{d}t'\,F(t')\,\chi_{\hat{B},\hat{A}}(t,t')\,, \tag{D.18}$$

with the retarded susceptibility

$$\chi_{\hat{B},\hat{A}}(t,t') = \frac{i}{\hbar}\langle[\hat{B}_I(t), \hat{A}_I(t')]\rangle_0\Theta(t-t')\,. \tag{D.19}$$

Specializing on harmonic time dependence of the perturbation,

$$F(t) = F_0\mathrm{e}^{-i(\omega+i\delta)t}, \tag{D.20}$$

one can also derive the formulation in frequency space,

$$\langle\hat{B}\rangle_F(\omega) = \langle\hat{B}\rangle_0 + \langle\langle\hat{B}, \hat{A}\rangle\rangle\,(\omega+i\delta)\,F(\omega) \tag{D.21}$$