using the definition

$$\langle\langle \hat{B}, \hat{A} \rangle\rangle (\omega + i\delta) \equiv \chi_{\hat{B},\hat{A}}(\omega + i\delta) = \frac{i}{\hbar} \int_0^\infty \mathrm{d}t \langle [\hat{B}_I(t), \hat{A}] \rangle_0 \, \mathrm{e}^{i(\omega + i\delta)t} \,. \tag{D.22}$$

Performing the trace for an eigenbasis of the unperturbed Hamiltonian, inserting the identity $1 = \sum_m |m\rangle\langle m|$ between the operators, and reordering terms one obtains the (thermal) spectral representation,

$$\begin{aligned} \langle\langle \hat{B}, \hat{A} \rangle\rangle (\omega + i\delta) &= \frac{i}{\hbar} \frac{1}{Z_0} \sum_{n,m} \mathrm{e}^{-\beta E_n} \int_0^\infty \mathrm{d}t \Big(\mathrm{e}^{it(\omega + i\delta + \frac{E_n - E_m}{\hbar})} \langle n|\hat{B}|m\rangle\langle m|\hat{A}|n\rangle \\ &\qquad - \mathrm{e}^{it(\omega + i\delta - \frac{E_n - E_m}{\hbar})} \langle n|\hat{A}|m\rangle\langle m|\hat{B}|n\rangle \Big) \qquad \text{(D.23)} \\ &= -\frac{1}{Z_0} \sum_{n,m} \frac{\langle n|\hat{B}|m\rangle\langle m|\hat{A}|n\rangle}{\hbar(\omega + i\delta) + E_n - E_m} \left(\mathrm{e}^{-\beta E_n} - \mathrm{e}^{-\beta E_m} \right) . \qquad \text{(D.24)} \end{aligned}$$

If we further assume that the spectrum of $\hat{H}_0$ is bounded, we find from (D.24) for the limiting behavior,

$$\langle\langle \hat{B}, \hat{A} \rangle\rangle (\omega + i\delta) \overset{\omega \to \infty}{\longrightarrow} -\frac{1}{\hbar\omega} \langle [\hat{B}, \hat{A}] \rangle_0 + \mathcal{O}\big(\frac{1}{\omega^2}\big) \,. \tag{D.25}$$

For the usual case $\hat{A} = \hat{B}$ the leading term in (D.25) vanishes so that the susceptibility $\chi_{\hat{B},\hat{B}}$ then falls off at least as $1/\omega^2$. Finally, the imaginary part of the susceptibility can be simplified when $\langle n|\hat{B}|m\rangle\langle m|\hat{A}|n\rangle$ is real. In particular, one obtains for $\hat{B} = \hat{A}^\dagger$ (i.e., for $\hat{B} = \hat{A}$ when $\hat{A}$ is hermitian),

$$\begin{aligned} \mathrm{Im}\, \langle\langle \hat{B}^\dagger, \hat{B} \rangle\rangle (\omega) &= \frac{\pi}{\hbar Z_0} \sum_{n,m} \big| \langle m|\hat{B}|n\rangle \big|^2 \delta(\omega + \frac{E_n - E_m}{\hbar}) \big(\mathrm{e}^{-\beta E_n} - \mathrm{e}^{-\beta E_m} \big) \qquad \text{(D.26)} \\ &\overset{T \to 0}{\longrightarrow} \frac{\pi}{\hbar} \sum_m \big| \langle m|\hat{B}|0\rangle \big|^2 \big(\delta(\omega - \frac{E_m - E_0}{\hbar}) - \delta(\omega + \frac{E_m - E_0}{\hbar}) \big) \,. \qquad \text{(D.27)} \end{aligned}$$