Why is $H_i(E,\mu)$ poisson distributed?

We assume a large number $N_0$ $(N_0 \to \infty)$ of measurements $n$ of the random variable $[H_i(E\mu)](n) \in \mathbb{N}_0$

Let $p(n)$ be the probability to find a value $n$

Maximum entropy determination of $p(n)$:

maximize $S = -\sum_n p(n) \ln(p_n)$ (Shannon entropy)

under the constraints

$$\sum_n p(n) = 1$$

$$\sum_n n\, p(n) = \overline{H_i(E,\mu)} = \overline{H_i}$$

$$\delta_{p(k)}\left[-\sum_n p(n)\ln p(n) - \alpha\left(\sum_n p(n) - 1\right) - \beta\left(\sum_n n\,p(n) - \overline{H_i}\right)\right] = 0$$

$$-\ln p(k) - 1 - \alpha - \beta k = 0$$

$$\ln p(k) = -\alpha - 1 - \beta k$$

$$p(k) = e^{-(\alpha+1)} e^{-\beta k} \qquad \beta > 0$$

Normalization: $\sum_{k=0}^{\infty} p(k) = 1 = e^{-(1+\alpha)} \sum_{k=0}^{\infty} e^{-|\beta| k}$

$$\approx e^{-(1+\alpha)} \int_0^{\infty} dk\, e^{-|\beta| k}$$

$$= \frac{e^{-(1+\alpha)}}{|\beta|} \Rightarrow |\beta| = e^{-(1+\alpha)}$$

$$\to p(k) = |\beta|\, e^{-|\beta| k}$$

First moments: $\sum_k k\, p(k) = \overline{H_i} = \sum_{k=0}^{\infty} |\beta|\, k\, e^{-|\beta| k}$

$$\approx \frac{1}{|\beta|} \int_0^{\infty} dx\, x\, e^{-x}$$

$$= \frac{1}{|\beta|}\left\{-x e^{-x}\Big|_0^{\infty} + \int_0^{\infty} e^{-x} dx\right\} = \frac{1}{|\beta|} = \overline{H_i}$$

$$\Rightarrow \quad \boxed{p(H_i(E,u)) = \frac{1}{\overline{H_i(E,u)}}\, e^{-H_i(E,u)/\overline{H_i(E,u)}}}$$

Exponential distribution

What is $\overline{H_i^2(E,u)}$

$$p(x) = \frac{1}{\bar{x}}\, e^{-\frac{x}{\bar{x}}} \; ; \quad \bar{x} = \int_0^\infty dx\, \frac{x}{\bar{x}}\, e^{-\frac{x}{\bar{x}}} = \bar{x} \int_0^\infty du\, u\, e^{-u} = \bar{x}$$

$$\overline{x^2} = \int_0^\infty dx\, \frac{x^2}{\bar{x}}\, e^{-x/\bar{x}} = \bar{x}^2 \int_0^\infty du\, u^2 e^{-u} = \bar{x}^2\, \Gamma(3) = 2\bar{x}^2$$

$$\Rightarrow \quad \overline{\delta x^2} = \bar{x}^2$$

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BUT, $k$ is limited by $N_i$ &

$$\Rightarrow \quad \sum_{k=0}^{N_i} p(k) = 1 \qquad \&$$

$$1 = e^{-(1+\alpha)} \sum_{k=0}^{N_i} e^{-|\beta| k} = e^{-(1+\alpha)}\, \frac{1 - e^{-|\beta|(N+1)}}{1 - e^{-|\beta|}}$$

$$\overline{H_i} = \frac{1 - e^{-|\beta|}}{1 - e^{-|\beta|(N+1)}} \sum_{k=0}^{N} k\, e^{-|\beta| k}$$

$$= \frac{1 - e^{-|\beta|}}{1 - e^{-|\beta|(N+1)}} \left(-\frac{\partial}{\partial \beta}\right) \sum_{k=1}^{N} e^{-|\beta| k}$$

$$= \frac{1 - e^{-|\beta|}}{1 - e^{-|\beta|(N+1)}} \left(-\frac{\partial}{\partial \beta}\right) \left( \frac{1 - e^{-|\beta|(N+1)}}{1 - e^{-|\beta|}} - 1 \right)$$

$$= -\frac{(N+1)\, e^{-|\beta|(N+1)}}{1 - e^{-|\beta|(N+1)}} + \frac{1}{1 - e^{-|\beta|(N+1)}}\, \frac{1}{1 - e^{-|\beta|}}\, |\beta|\, e^{-|\beta|}$$

Now consider the limit of large $N_i$

$$e^{-(1+\alpha)} = \left(1 - e^{-|\beta|}\right)$$

$$\overline{H_i} = \frac{|\beta|\, e^{-|\beta|}}{1 - e^{-|\beta|}}$$